A Survey On Mixed Finite Element Approximations

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Abstract **A brief discussion on mixed finite element formulations is carried out. Examples of approximations including the so called "face" and "edge" elements are given, together with abstract results and remarks on computational aspects.**

I. Mixed Formulations

The use of mixed formulations is becoming increasingly popular in many applications such as structural mechanics, fluid mechanics and, more recently, electrical engineering. A mixed formulation is typically obtained by factoring the equations to be solved into a system of first order equations, which are then cast into variational form and discretized by the finite element method. In fact, first order systems often arise directly in physical models, for example, as the constitutive and equilibrium laws in elasticity or as Maxwell's equations in electromagnetism. To illustrate the basic features of the method, let us consider the simple problem of the Poisson equation for the electrostatic potential V:

$$\begin{cases} -\mathrm{div}\,(\epsilon \underline{\nabla} V) = \rho & \text{in } \Omega \subset \mathbb{R}^n, \\ V = V_a & \text{on } \Gamma_a, \\ \dfrac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega\backslash\Gamma_a. \end{cases} \tag{1}$$

The first step towards a mixed formulation of (1) is to bring it back to a first order system

$$\underline{D} = -\epsilon \underline{\nabla} V \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{2}$$

$$\mathrm{div}\,\underline{D} = \rho \qquad \text{in } \Omega \subset \mathbb{R}^n, \tag{3}$$

with the boundary conditions:

$$V = V_a \qquad \text{on } \Gamma_a, \tag{4}$$

$$\underline{D}\cdot\underline{\nu} = 0 \qquad \text{on } \partial\Omega\backslash\Gamma_a. \tag{5}$$

To derive a variational formulation for (2)-(3) we multiply (2) by a test vector $\delta\underline{D}$ satisfying (5), and integrate over Ω. Then, integration by parts of the right-hand side leads to:

$$\int_\Omega (\epsilon^{-1}\underline{D})\cdot(\delta\underline{D})\,dx = \int_\Omega V\,\mathrm{div}\,(\delta\underline{D})\,dx - \int_{\Gamma_a} V_a(\delta\underline{D})\cdot\underline{\nu}\,ds \quad \forall\delta\underline{D}. \tag{6}$$

Similarly, we multiply (3) by a test function δV and integrate over Ω:

$$\int_\Omega (\mathrm{div}\,\underline{D})\delta V\,dx = \int_\Omega \rho\delta V\,dx \quad \forall\delta V. \tag{7}$$

From the mathematical point of view it is now clear from (6)-(7) that the natural spaces to look for a solution ($\underline{D}$, V) of (6)-(7) are the following:

$$V \in L^2(\Omega), \tag{8}$$

$$\underline{D} \in W := \{\underline{\tau} \in H(\mathrm{div};\Omega)\ \ \underline{\tau}\cdot\underline{\nu} = 0 \text{ on } \partial\Omega\backslash\Gamma_a\}, \tag{9}$$

where

$$H(\mathrm{div};\Omega) = \{\underline{\tau} \in (L^2(\Omega))^n,\ \mathrm{div}\,\underline{\tau} \in L^2(\Omega)\}. \tag{10}$$

We see from (6)-(7) two main features of mixed formulations. First, they typically involve two or more fields (here $\underline{D}$ and V). A general abstract result for two (or more)-field formulations will be given in the third section. Second, mixed formulations often require the use of "non traditional" spaces such as $H(\mathrm{div};\Omega)$ (see (10)), $H(\mathrm{curl};\Omega)$, or $H(\underline{\mathrm{curl}};\Omega)$, although such spaces (and their approximations) may be useful in more general situations, even without mixed (two-field) formulations.

II. Face Elements And Generalizations

When constructing finite element subspaces of W and of $L^2(\Omega)$, no continuity is required on V, while $\underline{D} \in W$ implies that the normal component of $\underline{D}$ must be continuous across interelement boundaries. Hence, finite element approximations of the mixed formulation (6)-(7) require the use of piecewise polynomial vectors having normal components continuous across interelement boundaries. Many families of mixed finite elements satisfying this property have been introduced and analyzed

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[3], [9]. We refer to [4] for a detailed treatment and exhaustive literature. Let us see few examples. First, let us construct the lowest order Raviart-Thomas element for a 2-D problem [9]. Let $\mathcal{T}_h$ be a regular family of decompositions of Ω into triangles T [5]. Define our finite element spaces $W_h \subset W$ and $M_h \subset L^2(\Omega)$ as

$$W_h = \{\underline{\tau} \in W : \underline{\tau}_{|\mathrm{T}} = (a_1 + bx, a_2 + by) \\ (= \underline{a} + b\underline{r}),\ a_1, a_2, b \in \mathbb{R}\} \quad (11)$$

$$M_h = \{v : v \text{ piecewise constant}\}. \quad (12)$$

Notice that a vector in W_h has normal component which is constant on each edge of each triangle. Hence, as degrees of freedom, the value of the normal component on each edge can be taken. When dealing with 3-D problems, finite element discretizations of $H(\mathrm{div};\Omega)$ can be easily constructed as natural generalization of (11). For a given decomposition of Ω into tetrahedrons, we take vectors having the form $(a_1 + bx, a_2 + by, a_3 + bz)(= \underline{a} + b\underline{r})$ locally, and we use the (constant) values of the normal components on the four faces as degrees of freedom. For this reason these elements are often referred to as "face" elements.

Let us see now examples of higher order elements. We consider first the lowest order BDM-element [3], where the vector space W_h is given by

$$W_h = \{\underline{\tau} \in W : \underline{\tau}_{|\mathrm{T}} \in \underline{P}_1(\mathrm{T})\}, \quad (13)$$

with $P_1(\mathrm{T})$ = polynomials of degree ≤ 1 on T. The scalar space M_h is the same as in (12). As degrees of freedom in W_h we take the values of the normal component at two different points on each edge ($3 \times 2 = 6$ d.o.f.). The extension of this element to 3-D problems uses vectors which are locally $\underline{P}_1$; the corresponding degrees of freedom are the values of the normal component at three different (not aligned) points on each face ($3 \times 4 = 12$ d.o.f.). The scalar variable is always piecewise constant. Finally, the "next to the lowest" Raviart-Thomas element uses the following choice:

$$W_h = \{\underline{\tau} \in W : \underline{\tau}_{|\mathrm{T}} \in \underline{P}_1(\mathrm{T}) + \underline{r}\widetilde{P}_1(\mathrm{T})\}, \quad (14)$$

$$M_h = \{v \in L^2(\Omega) : v_{|\mathrm{T}} \in P_1(\mathrm{T})\}. \quad (15)$$

In (14) $\widetilde{P}_1(\mathrm{T})$ denotes the set of homogeneous polynomials of degree 1. The degrees of freedom for the vectors in (14) can be taken as: the values of the normal component at two different points on each edge plus the meanvalue of each component on T ($3 \times 2 + 2 = 8$ d.o.f.). In 3-D, the definitions of W_h and M_h are formally identical to (14)-(15). The degrees of freedom for (14) will be the values of the normal component at three different (not aligned) points on each face, plus the meanvalue of each component on the tetrahedron ($3 \times 4 + 3 = 15$ d.o.f.).

III. Abstract Treatment Of Two-field Formulations

Problems of the type (6)-(7) enter a more general framework that reads as follows [2]: let Σ, U be Hilbert spaces, and consider the problem

$$\begin{cases} \text{find } (\underline{\sigma},\, u) \in \Sigma \times U \text{ such that} \\ a(\underline{\sigma}, \underline{\tau}) \ +\ b(\underline{\tau}, u) \ = < g, \underline{\tau} > \quad \forall \underline{\tau} \in \Sigma, \\ b(\underline{\sigma}, v) \ = < f, v > \quad \forall v \in U, \end{cases} \quad (16)$$

where a and b are bilinear continuous forms on $\Sigma \times \Sigma$ and $\Sigma \times U$ respectively, g and f are linear continuous functionals on Σ and U respectively. Sufficient conditions in order to ensure existence and uniqueness of the solution of (16) for all g and f (with continuous dependence) are the following [2]:

$$\begin{cases} \exists \alpha > 0 \text{ such that} \\ a(\underline{\tau}, \underline{\tau}) \ \geq\ \alpha ||\underline{\tau}||^2_\Sigma \quad \forall \underline{\tau} \in K, \end{cases} \quad (17)$$

where

$$K \ =\ \{\underline{\tau} \in \Sigma \ :\ b(\underline{\tau}, v) \ =\ 0 \ \ \forall v \in U\}, \quad (18)$$

and

$$\begin{cases} \exists \beta > 0 \text{ such that} \\ \displaystyle\operatorname*{Sup}_{\underline{\tau} \in \Sigma} \frac{b(\underline{\tau}, v)}{||\underline{\tau}||_\Sigma} \geq \beta ||v||_U \quad \forall v \in U. \end{cases} \quad (19)$$

When discretizing problem (16) we take finite dimensional subspaces $\Sigma_h \subset \Sigma$ and $U_h \subset U$, and we consider the discretized problem

$$\begin{cases} \text{find } (\underline{\sigma}_h,\, u_h) \in \Sigma_h \times U_h \text{ such that} \\ a(\underline{\sigma}_h, \underline{\tau}) \ +\ b(\underline{\tau}, u_h) \ = < g, \underline{\tau} > \quad \forall \underline{\tau} \in \Sigma_h, \\ b(\underline{\sigma}_h, v) \ = < f, v > \quad \forall v \in U_h. \end{cases} \quad (20)$$

Sufficient conditions in order to guarantee existence and uniqueness of the solution of (20) for all g and f and optimal error bounds are the following [2]:

$$\begin{cases} \exists \tilde{\alpha} > 0 \text{ independent of h such that} \\ a(\underline{\tau}, \underline{\tau}) \ \geq\ \tilde{\alpha} ||\underline{\tau}||^2_\Sigma \quad \forall \underline{\tau} \in K_h, \end{cases} \quad (21)$$

where

$$K_h \ =\ \{\underline{\tau} \in \Sigma_h \ :\ b(\underline{\tau}, v) \ =\ 0 \ \ \forall v \in U_h\}, \quad (22)$$

and

$$\begin{cases} \exists\widetilde{\beta} > 0 \text{ independent of h such that} \\ \displaystyle\operatorname*{Sup}_{\underline{\tau}\in\Sigma_h} \frac{b(\underline{\tau}, v)}{||\underline{\tau}||_\Sigma} \geq \widetilde{\beta}||v||_U \quad \forall v \in U_h. \end{cases} \tag{23}$$

More precisely, we have the following result: if conditions (22)-(23) are verified, problem (20) has a unique solution and the following error bound holds

$$||\underline{\sigma} - \underline{\sigma}_h||_\Sigma + ||u - u_h||_U \leq \gamma\{\inf_{\underline{\tau}\in\Sigma_h} ||\underline{\sigma} - \underline{\tau}||_\Sigma + \inf_{v_h\in U_h} ||u - v_h||_U\} \tag{24}$$

with γ independent of h.

In practice, conditions (21)-(23) are difficult to enforce and to verify. However, the following sufficient conditions are often useful. Assume that U is $L^2(\Omega)$ (or a power of it). Let B be the linear operator from Σ to U associated with the bilinear form b, that is, $b(\underline{\tau}, v) = (B\underline{\tau}, v)$ for all $\underline{\tau} \in \Sigma$ and $v \in U$. In the applications, B will be the *div* (or the *curl*) operator. The first assumption to be made is that

$$B \text{ maps } \Sigma_h \text{ into } U_h. \tag{25}$$

Let P_h be the orthogonal projection from U onto U_h; assume that there exists a linear continuous operator Π_h from Σ onto Σ_h with norm bounded independent of h such that, for all $\underline{\sigma} \in \Sigma$, we have

$$B(\Pi_h\underline{\sigma}) = P_h B\underline{\sigma}. \tag{26}$$

We have now the following result. If (17)-(19) hold, and (25)-(26) hold, then (21)-(23) are verified with $\widetilde{\alpha}$ and $\widetilde{\beta}$ independent of h.

In the applications that we have seen above, namely (6)-(10), the abstract spaces Σ and U are $H(\text{div};\Omega)$ and $L^2(\Omega)$ respectively. Accordingly, the abstract finite dimensional subspaces Σ_h and U_h are W_h and M_h. The operator B is the divergence operator, and it is immediate to check that it maps W_h into M_h for all the above choices. Similarly, the operator Π_h can be constructed easily, essentially interpolating the natural degrees of freedom. Care should be taken only in substituting point values with appropriate averages.

IV. Edge Elements And Generalizations

In the electromagnetic context, instead of continuity of the type $H(\text{div};\Omega)$ required by (9), continuity of the type $H(\text{curl};\Omega)$ or $H(\underline{\text{curl}};\Omega)$ is often required. We recall that, in 2-D, $\text{curl}\,\underline{\tau}$ is a scalar given by $\text{curl}\,\underline{\tau} = \frac{\partial\tau_2}{\partial x} - \frac{\partial\tau_1}{\partial y}$ and $H(\text{curl};\Omega)$ is the set of 2-vectors $\underline{\tau} \in (L^2(\Omega))^2$ such that $\text{curl}\,\underline{\tau} \in L^2(\Omega)$. On the other hand, in 3-D, $H(\underline{\text{curl}};\Omega)$ is the set of 3-vectors $\underline{\tau} \in (L^2(\Omega))^3$ such that $\underline{\text{curl}}\,\underline{\tau} \in (L^2(\Omega))^3$. This implies that, when constructing finite element subspaces of $H(\text{curl};\Omega)$, or $H(\underline{\text{curl}};\Omega)$, piecewise polynomial vectors having tangential component continuous at the interelement boundaries have to be used. Clearly, in 2-D problems the spaces $H(\text{div};\Omega)$ and $H(\text{curl};\Omega)$ coincide, up to a $\pi/2$ rotation, so that finite elements as (11)-(12) can be used, (or (13)-(12), or (14)-(15)). In particular, for instance, (11) has to be changed so that W_h is made of piecewise polynomial vectors locally having the form $(a_1 + by, a_2 - bx)$. Accordingly, the values of the tangential component on the edges have to be used as degrees of freedom. The choice (13) for W_h can be left unchanged, but we have to use tangential components instead of normals in the definition of the degrees of freedom. Similarly, in the definition (14), it is sufficient to substitute $\underline{r} = (x, y)$ with $\underline{r}^* = (y, -x)$, and, again, to use tangential instead of normal components in the degrees of freedom. This similarity between the $H(\text{div};\Omega)$ and the $H(\text{curl};\Omega)$ space stops holding for 3-D problems, and different finite element discretizations of $H(\underline{\text{curl}};\Omega)$ have to be constructed in order to preserve continuity of the tangential components across interelement boundaries [7], [8]. For instance, the lowest order element is made of vectors having the local form

$$\begin{aligned} \underline{\tau}_{|\mathrm{T}} &= (a_1 - b_3y + b_2z, a_2 + b_3x - b_1z, a_3 + b_2x + b_1y) \\ &\equiv (\underline{a} + \underline{b} \times \underline{r}). \end{aligned} \tag{27}$$

It can be easily checked [7] that: 1) the tangential component of vectors of the form (27) is constant on a straight line; 2) if two vectors of the form (27) have the same tangential component on each edge of a given triangular face, then the tangential componenents coincide on the whole face. Therefore, for piecewise polynomial vectors locally having the form (27), it is appropriate to take the (constant) values of the tangential component on the six edges of each tetrahedron as degrees of freedom. This is the reason why these elements are often referred to as "edge" elements.

Let us see shortly the local form and the degrees of freedom of vector spaces analogues to the choices (13) and (14). The space (13) remains formally unchanged. The choice of degrees of freedom now becomes the value of the tangential component at two different points on each edge. The $H(\underline{\text{curl}};\Omega)$ analog of (14) is slightly more delicate. We now take

$$\underline{\tau}_{|\mathrm{T}} \in \underline{P}_1(\mathrm{T}) + \underline{r} \times \underline{\widetilde{P}}_1(\mathrm{T}), \tag{28}$$

but we have to notice that the local space (28) has dimension 20 instead of the apparent 21 ($= 4\times3+3\times3$). The reason for this is that $\underline{\tilde{P}}_1$ contains $\underline{r}$, and $\underline{r}\times\underline{r}=0$. As degrees of freedom we now take the values of the tangential components at two different points on each edge (12 d.o.f.), plus the average of the tangential components on each face (8 d.o.f.).

V. Computational Aspects

Let us now turn to the numerical treatment of a mixed formulation, concentrating, for simplicity, on face elements (11)-(12). After discretization, problem (6)-(7) can be written in matrix form as

$$\begin{pmatrix} \tilde{A} & -\tilde{B} \\ \tilde{B}^* & 0 \end{pmatrix} \begin{pmatrix} \underline{D} \\ V \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix} \tag{29}$$

with obvious notation. The problem with system (29) is that the matrix is indefinite. However, this inconvenience can be eliminated by introducing Lagrange multipliers at the interfaces to relax the continuity required (on the normal or the tangential component, according to the problem we are dealing with). This amounts to using piecewise polynomial vectors of the form given in (11) but <u>discontinuous</u> at the interelement boundaries. The scalars will be the same as in (12), and a third finite dimensional space for the multipliers has to be introduced. In the case of our example, this space will be made of functions piecewise constant on each edge. This leads to a new problem which can be written in matrix form as

$$\begin{pmatrix} A & -B & C \\ B^* & 0 & 0 \\ C^* & 0 & 0 \end{pmatrix} \begin{pmatrix} \underline{D} \\ V \\ \Lambda \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \\ 0 \end{pmatrix} \tag{30}$$

The advantage in (30) is that matrix A is now a block diagonal matrix easy to invert element by element. Hence, $\underline{D}$ can be eliminated by static condensation, and (30) gives

$$\begin{pmatrix} B^*A^{-1}B & -B^*A^{-1}C \\ C^*A^{-1}B & -C^*A^{-1}C \end{pmatrix} \begin{pmatrix} V \\ \Lambda \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} \tag{31}$$

The matrix $B^*A^{-1}B$ is also block diagonal. Static condensation on V leads then to a final system, in the unknowns Λ only, whose matrix is symmetric and positive definite. This trick, first introduced in [6] for computational purposes, was studied theoretically in [1] where it was shown that the Lagrange multipliers actually provide better accuracy on the scalar variable (the potential V in our case) than that obtained directly.

Another remedy commonly used to solve problems of the form (29) is the following, often referred to as penalty method. Let λ be a "small" perturbation parameter, and let us modify equation (3) into

$$\operatorname{div}\underline{D} \;=\; \rho \;-\; \lambda V \qquad\qquad \text{in } \Omega\subset \mathbb{R}^n, \tag{32}$$

(V being the potential), so that the discretization of the perturbed problem leads to the following system (analogue of (29))

$$\begin{pmatrix} \tilde{A} & -\tilde{B} \\ \tilde{B}^* & \lambda\tilde{I} \end{pmatrix} \begin{pmatrix} \underline{D} \\ V \end{pmatrix} = \begin{pmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{pmatrix} \tag{33}$$

where $\tilde{I}$ is an approximation of the identity matrix (and can, in general, be taken diagonal with a suitable numerical integration). The second equation of (33) can then be explicitely solved for V, giving

$$V \;=\; \lambda^{-1}\tilde{I}^{-1}(\tilde{F}_2 \;-\; \tilde{B}^*\underline{D}). \tag{34}$$

Substituting (34) into the first equation of (33) yields

$$\tilde{A}\,\underline{D} \;+\; \lambda^{-1}\tilde{B}\tilde{I}^{-1}\tilde{B}^*\underline{D} \;=\; \tilde{F}_1 \;+\; \lambda^{-1}\tilde{I}^{-1}\tilde{F}_2, \tag{35}$$

that is, a linear system in the unknown $\underline{D}$ with a symmetric and positive definite matrix. It can be shown that, if the discretization has been properly made (that is, in accordance with the general rules (21)-(23)), the perturbation in the solution is of the order of λ. This is not true, in general, if conditions (21)-(23) are violated: the perturbed problem is solvable for every $\lambda > 0$ but the solution can degenerate when λ tends to zero.

VI. Conclusions

In the formulation of the basic laws of electromagnetism as a first order system (as in the classical Maxwell's equations) one has to deal with vector fields that need to have only some components continuous (normal or tangential). The most suitable spaces to set these preblems are therefore spaces of the type $H(\text{div};\Omega)$ and $H(\text{curl};\Omega)$ (or $H(\underline{\text{curl}};\Omega)$). Several ways for approximating these spaces have been presented above. The use of these formulations (and of the corresponding discretizations) often leads to indefinite systems with indefinite matrices of the form

$$\begin{pmatrix} A & -B \\ B^* & 0 \end{pmatrix}. \tag{36}$$

Conditions for well-posedness of systems of this type have been recalled, together with examples of their applications to non-classical finite element spaces in electromagnetic contexts. Algorithms and tricks for the numerical solution of such indefinite systems have been presented.

In other contexts (structures, fluids, etc.) mixed formulations, once well understood, have proved to be a very powerful tool, often providing at the same time a better accuracy and a better fulfilment of the basic physical laws. This encourages in pursuing their analysis and experimentation for electromagnetic problems.

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A list of additional references on the subject, not quoted directly in the text, is given in the following section for the convenience of the reader.

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