A nonconforming element for the Reissner–Mindlin plate

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Abstract

We develop a locking free nonconforming element for the Reissner-Mindlin plate using Discontinuous Galerkin techniques, and prove optimal error estimates.

Key words: Reissner-Mindlin plates, Discontinuous Finite Element Methods.

1 Introduction

In recent times, there has been a considerable interest, mostly among mathematicians, in the extension of Discontinuous Galerkin methods to the treatment of elliptic problems (see, for instance, [5] and the references therein). Although their practical interest is still under investigation, it is clear that the DG approach often implies a different approach to the problem, that can sometimes lead, in the end, to new conforming or nonconforming finite elements that would have been more difficult to discover starting with the classical approach. This is surely the case, for instance, of the extension of the Crouzeix-Raviart element for Stokes problem or nearly incompressible elasticity problems (see [23]), or the higher order Arnold-Falk elements for Reissner-Mindlin plates (see [6]). The element that we are going to present here, again for Reissner-Mindlin plates, could be considered as another example in this direction. In a sense, being a nonconforming element, it could have been obtained directly with the more standard finite element machinery. However,

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the possibility of using such an element became clear only after using a DG approach.

The element, in essence, is based on the use of nonconforming piecewise linear functions for both rotations and transversal displacements. Thus, all the unknowns share the same nodes. For the element to work, however, we have to add some internal degrees of freedom (that could easily be eliminated by static condensation). There are many variants available for these internal degrees of freedom: here the whole discussion is made assuming that we have an additional nonconforming P_2-bubble (in barycentric coordinates, $\chi_2 := 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2$) added to each component of rotations, and to transversal displacements. Several possible variants are discussed, at the end, in Section 5.

We are fully aware that the main interest, for new Reissner-Mindlin elements, relies in the possibility of obtaining a convenient shell element out of them. Indeed, there are, by now, several elements that could be considered as satisfactory for the plate Reissner-Mindlin problem (see, just to name a few of them, [7] - [10], [13]. [15], [21], [24] - [26], and the references therein), but very few elements for shells have been analyzed in a thorough way, mathematically and experimentally (see, e.g. [3], [16] - [20], [22]). This, however, will not be discussed here, and will possibly be object of future works. We refer for instance to [12]-[19] for a wider discussion and more references.

The promising features of this element are its semplicity, the low degree, and, as already pointed out, the fact that all variables share the same nodes (the midpoints of the edges). The element has optimal order af approximation and is locking free. Compared with an ideal *conforming* linear element, we have here more degrees of freedom for the same mesh. However, in several experiments on various types of elliptic problems, the ratio *accuracy/d.o.f.* for conforming and nonconforming linear elements turned out to be quite similar (the formers having a slight edge in the presence of very regular solutions, the latters being preferable for less regular ones).

Hence, all together, we believe that the extension of such an element to shell problems has, at least, good possibilities.

An outline of the paper is as follows. In Section 2 we present the problem and recall some of the difficulties related to the numerical treatment. In Section 3 we introduce the nonconforming element, based on typical instruments of Discontinuous Galerkin approach. In Section 4 we prove error estimates. Finally, in Section 3 we show possible variants of the element discussed .

2 The problem

Given g in, say, $L^2(\Omega)$, the Reissner–Mindlin equations with clamped boundary require to find $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ such that

$$-\operatorname{div}\mathbf{C}\,\varepsilon(\boldsymbol{\theta}) - \boldsymbol{\gamma} = 0 \quad \text{in } \Omega, \tag{1}$$

$$-\operatorname{div}\boldsymbol{\gamma} = g \quad \text{in } \Omega, \tag{2}$$

$$\boldsymbol{\gamma} = \lambda t^{-2}(\boldsymbol{\nabla} w - \boldsymbol{\theta}) \quad \text{in } \Omega, \tag{3}$$

$$\boldsymbol{\theta} = 0,\ w = 0 \text{ on } \partial\Omega. \tag{4}$$

In (1)-(3), $\mathbf{C}$ is the tensor of bending moduli, $\boldsymbol{\theta}$ represents the rotations, w the transversal displacement, and $\boldsymbol{\gamma}$ the scaled shear stresses. Moreover, ε is the usual symmetric gradient operator, $\lambda(= 5/6)$ is the shear correction factor, and t is the thickness.

The above equations correspond to the minimization of the functional

$$J^t(\boldsymbol{\eta}, v) = \frac{1}{2}a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{\lambda t^{-2}}{2}||\boldsymbol{\nabla} v - \boldsymbol{\eta}||^2_{0,\Omega} - (g, v), \tag{5}$$

where

$$a(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_\Omega \mathbf{C}\,\varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta})\mathrm{d}x,$$

and $(\cdot,\cdot)$ (resp. $||\cdot||_{0,\Omega}$) is the inner-product (resp. norm) in $L^2(\Omega)$. The classical variational formulations of problem (1)–(3) is

$$\begin{cases} \text{Find } (\boldsymbol{\theta}, w, \boldsymbol{\gamma}) \in (H^1_0(\Omega))^2 \times H^1_0(\Omega) \times (L^2(\Omega))^2 : & \\ a(\boldsymbol{\theta}, \boldsymbol{\eta}) - (\boldsymbol{\gamma}, \boldsymbol{\eta}) = 0 & \boldsymbol{\eta} \in (H^1_0(\Omega))^2, \\ (\boldsymbol{\gamma}, \boldsymbol{\nabla} v) = (g, v) & v \in H^1_0(\Omega), \\ \dfrac{t^2}{\lambda}(\boldsymbol{\gamma}, \boldsymbol{\tau}) - (\boldsymbol{\nabla} w, \boldsymbol{\tau}) + (\boldsymbol{\theta}, \boldsymbol{\tau}) = 0 & \boldsymbol{\tau}(L^2(\Omega))^2. \end{cases} \tag{6}$$

It is known that, keeping g fixed, and letting $t \to 0$, the minimizing argument $(\boldsymbol{\theta}^t, w^t)$ of $J^t(\boldsymbol{\eta}, v)$ tends to a finite limit (θ^0, w^0) such that $\theta^0 = \nabla w^0$, and w^0 is the minimizing argument of $\frac{1}{2}a(\nabla v, \nabla v) - (g, v)$ over $H^2_0(\Omega)$ (that is, the solution of the Kirchhoff model; see, for instance, [14]).

A conforming approximation of the problem leads to introduce finite element subspaces $\Theta_h \subset (H^1_0(\Omega))^2$ and $W_h \subset H^1_0(\Omega)$, and to look for a pair $(\boldsymbol{\theta}^t_h, w^t_h)$ minimizing (5) over $\Theta_h \times W_h$. It is expected that, for h small, the sequence of solutions tend, for $t \to 0$, to a limit (θ^0_h, w^0_h) close to (θ^0, w^0). Indeed, if this is not the case, then the convergence (in h) of (θ^t_h, w^t_h) to (θ^t, w^t) cannot be uniform in t, and this is a problem when $t << diam(\Omega)$.

On the other hand, it is clear that we must have

$$\theta_h^0 = \nabla w_h^0. \tag{7}$$

For simple-minded discretizations, it can occur that the set of pairs $(\theta_h, w_h) \in \Theta_h \times W_h$ satisfying (7) is very small.

For instance, if both Θ_h and W_h are made of piecewise linear continuous functions, then (7) implies $\theta_h^0 = \nabla w_h^0 = 0$. This is the locking phenomenon. In order to avoid locking, a typical remedy is to change J^t into

$$J_h^t(\eta, v) := \frac{1}{2} a(\boldsymbol{\eta}, \boldsymbol{\eta}) + \frac{\lambda t^{-2}}{2} ||P_h(\boldsymbol{\nabla} v - \boldsymbol{\eta})||_{0,\Omega} - (g, v), \tag{8}$$

where P_h is a suitable projection (or interpolation) operator, in general on some lower degree polynomials. In the engineering practice, the reduction corresponding to the use of P_h is actually often realized by using a reduced integration formula in the shear term.

3 Nonconforming approximation

We shall introduce a nonconforming finite element approximation of problem (1)–(3) using a Discontinuous Galerkin type approach. Let then $\mathcal{T}_h$ be a decomposition of Ω into triangles T. As we are going to work with discontinuous elements, the starting working space will be

$$H^1(\mathcal{T}_h) := \prod_{T \in \mathcal{T}_h} H^1(T) \qquad \text{with seminorm} \quad |v|_{1,h}^2 = \sum_{T \in \mathcal{T}_h} ||\boldsymbol{\nabla} v||_{0,\Omega}^2. \tag{9}$$

For vector valued functions we shall use $(H^1(\mathcal{T}_h))^2$, and for tensors $(H^1(\mathcal{T}_h))_s^4$. A typical instrument of the DG approach is the use of jumps and averages, that have to be defined. We denote by $\mathcal{E}_h$ the set of all the edges in $\mathcal{T}_h$, and by $\mathcal{E}_h'$ the set of internal edges. Let e be an internal edge of $\mathcal{T}_h$, shared by two elements E^+ and E^-, and let φ denote a function in $H^1(\mathcal{T}_h)$, or a vector in $(H^1(\mathcal{T}_h))^2$, or a tensor in $(H^1(\mathcal{T}_h))_s^4$. We define the average as usual:

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2} \quad \forall e \in \mathcal{E}_h'. \tag{10}$$

For a scalar function $\varphi \in H^1(\mathcal{T}_h)$ we define its jump as

$$[\varphi] = \varphi^+ \mathbf{n}^+ + \varphi^- \mathbf{n}^- \quad \forall e \in \mathcal{E}_h', \tag{11}$$

while the jump of a vector $\boldsymbol{\varphi} \in (H^1(\mathcal{T}_h))^2$ is given by:

$$[\boldsymbol{\varphi}] = (\boldsymbol{\varphi}^+ \otimes \mathbf{n}^+)_S + (\boldsymbol{\varphi}^- \otimes \mathbf{n}^-)_S \quad \forall e \in \mathcal{E}_h', \tag{12}$$

where $(\boldsymbol{\varphi} \otimes \mathbf{n})_S$ denotes the symmetric part of the tensor product, and $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to ∂E^+ (resp. to ∂E^-). We do not need to define jumps of tensors. On the boundary edges we define jumps of scalars as $[\varphi] = \varphi \mathbf{n}$, and jumps of vectors as $[\boldsymbol{\varphi}] = (\boldsymbol{\varphi} \otimes \mathbf{n})_S$, where $\mathbf{n}$ is the outward unit normal to $\partial\Omega$. We also define averages of vectors and tensors as $\{\boldsymbol{\varphi}\} = \boldsymbol{\varphi}$. It can be easily checked that, if $\boldsymbol{\varphi}$ is a smooth tensor, and $\boldsymbol{\eta}$ a piecewise smooth vector, the following equality holds (see, e.g., [4] for a similar computation):

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} \boldsymbol{\varphi} \mathbf{n} \cdot \boldsymbol{\eta} \, ds = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\varphi}\} : [\boldsymbol{\eta}] \, ds. \tag{13}$$

We now introduce the finite element spaces that we are going to use. On a generic triangle $T \in \mathcal{T}_h$ we define:

$$P(T) := P_1(T) \oplus \chi_2(T), \tag{14}$$

where $P_1(T)$ denotes the set of polynomials of degree ≤ 1 on T, and χ_2 denotes the nonconforming bubble of P_2, i.e., the polynomial of degree 2 vanishing at the two Gauss points of each edge. In barycentric coordinates this bubble has the expression (for instance),

$$\chi_2 = 3(\lambda_1^2 + \lambda_2^2 + \lambda_3^2) - 2. \tag{15}$$

We then define, locally, the finite element spaces for approximating $\boldsymbol{\theta}$, w, and $\boldsymbol{\gamma}$ as:

$$P_\theta(T) = (P_1(T))^2 \oplus \boldsymbol{\chi}_2(T), \tag{16}$$
$$P_w(T) = P_1(T) \oplus \chi_2(T) \tag{17}$$
$$P_\gamma(T) = (P_0(T))^2 \oplus \boldsymbol{\nabla}\chi_2(T). \tag{18}$$

(See figure 1 for the choice of degrees of freedom). Next, we form the finite

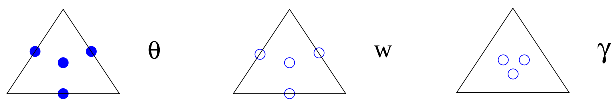

Fig. 1. Local dof for the three variables

element spaces:

$$\Theta_h = \{\boldsymbol{\eta} : \boldsymbol{\eta}_{|T} \in P_\theta(T), \ \int_e [\boldsymbol{\eta}] ds = 0 \ \forall e \in \mathcal{E}_h\}, \tag{19}$$

$$W_h = \{v : v_{|T} \in P_w(T), \ \int_e [v] ds = 0 \ \forall e \in \mathcal{E}_h\}, \tag{20}$$

$$\Gamma_h = \{\boldsymbol{\tau} : \boldsymbol{\tau}_{|T} \in P_\gamma(T)\}, \tag{21}$$

and notice that

$$\boldsymbol{\nabla}_h W_h \subset \Gamma_h, \tag{22}$$

where $\boldsymbol{\nabla}_h$ denotes the gradient element by element.

Proposition 1 *A vector $\boldsymbol{\tau} \in P_\gamma(T)$ is uniquely determined by the following 3 degrees of freedom:*

$$\int_T \boldsymbol{\tau}\, dx, \tag{23}$$

$$\int_T \operatorname{div} \boldsymbol{\tau}\, dx. \tag{24}$$

Proof Condition (24) determines the coefficient of the bubble part, while conditions (23) take care of the constant part of the components. □

The degrees of freedom (23)–(24) can be used to define the reduction operator $P_h : (H^1(\mathcal{T}_h))^2 \to \Gamma_h$.

Definition 2 *For any $\boldsymbol{\eta} \in (H^1(\mathcal{T}_h))^2$, $P_h\boldsymbol{\eta} \in \Gamma_h$ is defined locally by:*

$$\int_T (\boldsymbol{\eta} - P_h\boldsymbol{\eta})\, dx = 0 \quad \forall T \in \mathcal{T}_h, \tag{25}$$

$$\int_T \operatorname{div}(\boldsymbol{\eta} - P_h\boldsymbol{\eta})\, dx = 0 \quad \forall T \in \mathcal{T}_h. \tag{26}$$

It is easy to check that

$$\|P_h\boldsymbol{\eta}\|_{0,\Omega} \leq C\|\boldsymbol{\eta}\|_{0,\Omega} \quad \boldsymbol{\eta} \in \Theta_h. \tag{27}$$

Finally, we introduce a penalty on the jumps of functions in Θ_h as:

$$p_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e [\boldsymbol{\theta}] : [\boldsymbol{\eta}]\, ds, \tag{28}$$

and we define:

$$a_T(\boldsymbol{\theta}, \boldsymbol{\eta}) := \int_T \mathbf{C}\, \varepsilon(\boldsymbol{\theta}) : \varepsilon(\boldsymbol{\eta})\, dx. \tag{29}$$

The discrete problem is then

$$\begin{cases} \text{Find } (\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h) \in \Theta_h \times W_h \times \Gamma_h & \\ \displaystyle\sum_{T \in \mathcal{T}_h} a_T(\boldsymbol{\theta}_h, \boldsymbol{\eta}) + p_\Theta(\boldsymbol{\theta}_h, \boldsymbol{\eta}) - (\boldsymbol{\gamma}_h, P_h\boldsymbol{\eta}) = 0 & \boldsymbol{\eta} \in \Theta_h, \\ (\boldsymbol{\gamma}_h, \boldsymbol{\nabla}_h v) = (g, v) & v \in W_h, \\ \dfrac{t^2}{\lambda}(\boldsymbol{\gamma}_h, \boldsymbol{\tau}) - (\boldsymbol{\nabla}_h w_h, \boldsymbol{\tau}) + (P_h\boldsymbol{\theta}_h, \boldsymbol{\tau}) = 0 & \boldsymbol{\tau} \in \Gamma_h. \end{cases} \tag{30}$$

We point out that, since both $P_h\boldsymbol{\theta}_h$ and $\boldsymbol{\nabla}_h w_h$ belong to Γ_h, the third equation of (30) is just another way of writing

$$\boldsymbol{\gamma}_h = \lambda t^{-2}(\boldsymbol{\nabla}_h w_h - P_h\boldsymbol{\theta}_h). \tag{31}$$

Hence, $\boldsymbol{\gamma}_h$ can be eliminated elementwise, so that system (30) is the variational formulation of (8) in the unknowns $\boldsymbol{\theta}_h, w_h$ only. The introduction of the auxiliary variable $\boldsymbol{\gamma}_h$ is just a mathematical trick to perform the error analysis.

4 Error estimates

We shall prove error estimates in the following norms:

$$\|\boldsymbol{\eta}\|_{\Theta}^2 := \|\boldsymbol{\eta}\|_{0,\Omega}^2 + \|\varepsilon_h(\boldsymbol{\eta})\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|[\boldsymbol{\eta}]\|_{0,e}^2, \qquad \eta \in (H^1(\mathcal{T}_h))^2, \tag{32}$$

$$\|v\|_W^2 := \|v\|_{0,\Omega}^2 + |v|_{1,h}^2 =: \|v\|_{1,h}^2 \quad v \in H^1(\mathcal{T}_h), \tag{33}$$

$$\|\boldsymbol{\tau}\|_{\Gamma}^2 := \|\boldsymbol{\tau}\|_{0,\Omega}^2 \quad \boldsymbol{\tau} \in (H^1(\mathcal{T}_h))^2. \tag{34}$$

(In (32) ε_h denotes the symmetric gradient taken element by element). In the sequel we shall often use the following result (see [1]-[2]): let T be a triangle, and let e be an edge of T. Then $\exists C > 0$ only depending on the minimum angle of T such that

$$||\varphi||_{0,e}^2 \le C\Big(|e|^{-1}||\varphi||_{0,T}^2 + |e||\varphi|_{1,T}^2\Big) \qquad \varphi \in H^1(\mathcal{T}_h). \tag{35}$$

Clearly, (35) also holds for vector valued functions $\boldsymbol{\varphi} \in (H^1(\mathcal{T}_h))^2$.

Define:

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) := \sum_{T\in\mathcal{T}_h} a_T(\boldsymbol{\theta},\boldsymbol{\eta}) + p_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}), \quad \boldsymbol{\theta},\boldsymbol{\eta} \in (H^1(\mathcal{T}_h))^2, \tag{36}$$

and notice that

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) \le C\|\boldsymbol{\theta}\|_\Theta \|\boldsymbol{\eta}\|_\Theta \qquad \boldsymbol{\theta},\boldsymbol{\eta} \in (H^1(\mathcal{T}_h))^2, \tag{37}$$

$$a_h(\boldsymbol{\eta},\boldsymbol{\eta}) \ge \alpha\|\boldsymbol{\eta}\|_\Theta^2 \qquad \boldsymbol{\eta} \in \Theta_h. \tag{38}$$

We observe that the ellipticity property (38) is not trivial to prove. We refer for instance to [6], where the following result is proved:

$$\|\boldsymbol{\eta}\|_{0,\Omega}^2 \le C(\|\varepsilon_h(\boldsymbol{\eta})\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h} \frac{1}{|e|}\|[\boldsymbol{\eta}]\|_{0,e}^2), \qquad \boldsymbol{\eta} \in (H^1(\mathcal{T}_h))^2. \tag{39}$$

In (37), (39), and in the sequel we denote by C a positive constant independent of h, not necessarily the same at the various occurrencies.

Multiplying equation (1) by $\boldsymbol{\eta} \in \Theta_h$, integrating by parts, and using $[\boldsymbol{\theta}] = 0$ we obtain

$$a_h(\boldsymbol{\theta},\boldsymbol{\eta}) - (\boldsymbol{\gamma},\boldsymbol{\eta}) = c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) \qquad \boldsymbol{\eta} \in \Theta_h, \tag{40}$$

where, using (13),

$$c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\mathbf{n}\cdot\boldsymbol{\eta}\,ds = \sum_{e\in\mathcal{E}_h}\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\} : [\boldsymbol{\eta}]\,ds. \tag{41}$$

Multiplying equation (2) by $v \in W_h$ and integrating by parts we have

$$(\boldsymbol{\gamma},\boldsymbol{\nabla}_h v) = (g,v) + c_W(\boldsymbol{\gamma},v) \qquad v\in W_h, \tag{42}$$

where

$$c_W(\boldsymbol{\gamma},v) := \sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\gamma}\cdot\mathbf{n}v\,ds = \sum_{e\in\mathcal{E}_h}\int_e\{\gamma\}\cdot[v]\,ds. \tag{43}$$

Collecting (40), (42), and (3) we obtain

$$\begin{cases} a_h(\boldsymbol{\theta},\boldsymbol{\eta}) - (\boldsymbol{\gamma},\boldsymbol{\eta}) = c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) & \boldsymbol{\eta}\in\Theta_h,\\ (\boldsymbol{\gamma},\boldsymbol{\nabla}_h v) = (g,v) + c_W(\boldsymbol{\gamma},v) & v\in W_h,\\ \boldsymbol{\gamma} = \lambda t^{-2}(\boldsymbol{\nabla}w - \boldsymbol{\theta}). \end{cases} \tag{44}$$

By subtracting (30) from (44), and using (31) we can form the error equations

$$\begin{cases} a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\eta}) - (\boldsymbol{\gamma},\boldsymbol{\eta}) + (\boldsymbol{\gamma}_h,P_h\boldsymbol{\eta}) = c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) & \boldsymbol{\eta}\in\Theta_h,\\ (\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla}_h v) = c_W(\boldsymbol{\gamma},v) & v\in W_h,\\ \boldsymbol{\gamma}-\boldsymbol{\gamma}_h = \lambda t^{-2}[\boldsymbol{\nabla}_h(w-w_h) - (\boldsymbol{\theta}-P_h\boldsymbol{\theta}_h)]. \end{cases} \tag{45}$$

We see from (45) that the non conforming approach leads to consistency errors $c_W(\boldsymbol{\gamma},v)$ and $c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta})$ that need to be estimated. This can be done using (35) and the definition of the finite element spaces Θ_h, W_h, as shown in the following Proposition.

Proposition 3 *In the above assumptions, the consistency terms $c_W(\boldsymbol{\gamma},v)$ and $c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta})$ can be bound as:*

$$c_W(\boldsymbol{\gamma},v) \le Ch|\boldsymbol{\gamma}|_{1,\Omega}|v|_{1,h} \qquad v\in W_h, \tag{46}$$

$$c_\Theta(\boldsymbol{\theta},\boldsymbol{\eta}) \le Ch|\boldsymbol{\theta}|_{2,\Omega}\|\!|\boldsymbol{\eta}|\!\|_\Theta \qquad \boldsymbol{\eta}\in\Theta_h. \tag{47}$$

Proof Let $P_e^0(\boldsymbol{\gamma})$ and $P_e^0(\mathbf{C}\,\varepsilon(\boldsymbol{\theta}))$ denote constant approximations of $\boldsymbol{\gamma}$ and $\mathbf{C}\,\varepsilon(\boldsymbol{\theta})$ on e, respectively. Then, thanks to the definitions (19)-(20) of Θ_h and W_h, for every edge $e\in\mathcal{E}_h$ we have

$$\begin{aligned} &\int_e\{\boldsymbol{\gamma}\}\cdot[v]\,ds = \int_e\{\boldsymbol{\gamma}-P_e^0(\boldsymbol{\gamma})\}\cdot[v]\,ds \qquad \forall v\in W_h,\\ &\int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta})\} : [\boldsymbol{\eta}]\,ds = \int_e\{\mathbf{C}\,\varepsilon(\boldsymbol{\theta}) - P_e^0(\mathbf{C}\,\varepsilon(\boldsymbol{\theta}))\} : [\boldsymbol{\eta}]\,ds \qquad \forall\boldsymbol{\eta}\in\Theta_h. \end{aligned} \tag{48}$$

Using (48), Cauchy-Schwarz, (35), and classical interpolation results we obtain, for $v \in W_h$,

$$
\begin{aligned}
c_W(\boldsymbol{\gamma}, v) &= \sum_{e\in\mathcal{E}_h} \int_e \{\boldsymbol{\gamma} - P_e^0(\boldsymbol{\gamma})\} \cdot [v]\, ds \\
&\le \Big(\sum_{e\in\mathcal{E}_h} |e| ||\{\boldsymbol{\gamma} - P_e^0(\boldsymbol{\gamma})\}||_{0,e}^2\Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2\Big)^{1/2} \\
&\le C\Big(\sum_{T\in\mathcal{T}_h} (||\boldsymbol{\gamma} - P_e^0(\boldsymbol{\gamma})||_{0,T}^2 + |e|^2 |\boldsymbol{\gamma}|_{1,T}^2)\Big)^{1/2} \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2\Big)^{1/2} \\
&\le Ch|\boldsymbol{\gamma}|_{1,\Omega} \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2\Big)^{1/2}.
\end{aligned} \tag{49}
$$

Moreover, using similar arguments, always for $v \in W_h$ we have

$$
\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2 = \sum_{e\in\mathcal{E}_h} \frac{1}{|e|} \int_e [v - P_e^0(v)] \cdot [v]\, ds \le |v|_{1,h} (\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2)^{1/2}. \tag{50}
$$

Thus,

$$
(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[v]||_{0,e}^2)^{1/2} \le |v|_{1,h}, \qquad v \in W_h, \tag{51}
$$

and (46) follows. Proceeding in exactly the same way we obtain

$$
c_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) \le Ch|\boldsymbol{\theta}|_{2,\Omega} \Big(\sum_{e\in\mathcal{E}_h} \frac{1}{|e|} ||[\boldsymbol{\eta}]||_{0,e}^2\Big)^{1/2} \le Ch|\boldsymbol{\theta}|_{2,\Omega} |||\boldsymbol{\eta}|||_\Theta \qquad \boldsymbol{\eta} \in \Theta_h. \tag{52}
$$

□

We have now to introduce suitable interpolants of $\boldsymbol{\theta}$ and w.

Lemma 4 *For every $\boldsymbol{\theta} \in (H_0^1(\Omega))^2$, the following conditions*

$$
\int_e (\boldsymbol{\theta} - \boldsymbol{\theta}_I)\, ds = 0 \quad \forall e \text{ edge of } T, \quad \forall T, \tag{53}
$$

$$
\int_T (\boldsymbol{\theta} - \boldsymbol{\theta}_I)\, dx = 0 \quad \forall T, \tag{54}
$$

uniquely determine $\boldsymbol{\theta}_I \in \Theta_h$. Moreover, if $\boldsymbol{\theta} \in (H^2(\Omega))^2$, the following interpolation estimate holds:

$$
|||\boldsymbol{\theta} - \boldsymbol{\theta}_I|||_\Theta \le Ch|\boldsymbol{\theta}|_{2,\Omega}. \tag{55}
$$

Proof The 8 degrees of freedom (53)–(54) uniquely define $\boldsymbol{\theta}_I \in P_\theta(T)\ \forall T \in \mathcal{T}_h$. Indeed, $\boldsymbol{\chi}_{2|e}$ being a Legendre polynomial of degree 2, the 6 conditions (53) determine the linear part of $\boldsymbol{\theta}_I$, while the 2 remaining conditions (54) take care of the bubble part. Estimate (55) follows immediately from standard interpolation results, using (35) to estimate the jump terms. □

Lemma 5 *For every $w \in H^1_0(\Omega)$, the following conditions*

$$\int_e (w - w_I)\, ds = 0, \quad \forall e \text{ edge of } T, \quad \forall T, \tag{56}$$

$$\int_T \Delta(w - w_I)\, dx = 0 \quad \forall T, \tag{57}$$

uniquely determine $w_I \in W_h$. Moreover, if $w \in H^2(\Omega)$, the following interpolation estimate holds:

$$\|w - w_I\|_W \leq Ch|w|_{2,\Omega}. \tag{58}$$

Proof The 3 conditions (56) determine the linear part of w_I, and condition (57) takes care of the bubble part. Estimate (58) is a standard interpolation result. $\square$

Finally, it remains to define a suitable interpolant $\boldsymbol{\gamma}_I \in \Gamma_h$ of $\boldsymbol{\gamma}$. In analogy with the definition (31) for $\boldsymbol{\gamma}_h$, we define:

$$\boldsymbol{\gamma}_I = \lambda t^{-2}(\boldsymbol{\nabla}_h w_I - P_h \boldsymbol{\theta}_I). \tag{59}$$

The next Lemma provides a result that plays a crucial role for deriving error estimates.

Lemma 6 *Let $\boldsymbol{\gamma}_I \in \Gamma_h$ be defined as in* (59)*, where w_I is given in* (56)*-*(57)*, $\boldsymbol{\theta}_I$ in* (53)*-*(54)*, and $P_h\boldsymbol{\theta}_I$ is given by* (25)*-*(26)*. Then, the following fundamental property holds:*

$$\boldsymbol{\gamma}_I = P_h \boldsymbol{\gamma}. \tag{60}$$

Moreover, if $\boldsymbol{\gamma} \in (H^1(\Omega))^2$, the following interpolation estimate holds:

$$\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_I\|_\Gamma \leq Ch|\boldsymbol{\gamma}|_{1,\Omega}. \tag{61}$$

Proof By subtracting (59) from (3) we obtain

$$\lambda^{-1} t^2 (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I) = \boldsymbol{\nabla}_h (w - w_I) - (\boldsymbol{\theta} - P_h \boldsymbol{\theta}_I). \tag{62}$$

Hence, thanks to (56), (54), and (25),

$$\begin{aligned} \lambda^{-1} t^2 \int_T (\boldsymbol{\gamma} - \boldsymbol{\gamma}_I)\, dx &= \int_T \boldsymbol{\nabla}(w - w_I)\, dx - \int_T (\boldsymbol{\theta} - P_h \boldsymbol{\theta}_I)\, dx \\ &= -\int_T (\boldsymbol{\theta} - \boldsymbol{\theta}_I + \boldsymbol{\theta}_I - P_h \boldsymbol{\theta}_I)\, dx = 0, \end{aligned} \tag{63}$$

i.e., (25) is verified. Moreover, due to (57), (53), and (26),

$$\begin{aligned} \lambda^{-1} t^2 \int_T \operatorname{div}(\boldsymbol{\gamma} - \boldsymbol{\gamma}_I)\, dx &= \int_T \Delta(w - w_I)\, dx - \int_T \operatorname{div}(\boldsymbol{\theta} - P_h \boldsymbol{\theta}_I)\, dx \\ &= -\int_T \operatorname{div}(\boldsymbol{\theta} - \boldsymbol{\theta}_I + \boldsymbol{\theta}_I - P_h \boldsymbol{\theta}_I)\, dx = 0, \end{aligned} \tag{64}$$

i.e., (26) is also verified and the proof of (60) is concluded. Finally, (61) follows from standard interpolation results. □

We are now ready to prove the error estimates. Using (38)-(37) and the first equation in (45) we have:

$$\begin{aligned}\alpha|||\boldsymbol{\theta}_I-\boldsymbol{\theta}_h|||_\Theta^2 &\le a_h(\boldsymbol{\theta}_I-\boldsymbol{\theta},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)+a_h(\boldsymbol{\theta}-\boldsymbol{\theta}_h,\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)\\ &\le C|||\boldsymbol{\theta}-\boldsymbol{\theta}_I|||_\Theta|||\boldsymbol{\theta}_I-\boldsymbol{\theta}_h|||_\Theta+(\boldsymbol{\gamma},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)\\ &-(\boldsymbol{\gamma}_h,P_h(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h))+c_\Theta(\boldsymbol{\theta},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h).\end{aligned} \tag{65}$$

From (47) we have

$$c_\Theta(\boldsymbol{\theta},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)\le Ch|\boldsymbol{\theta}|_{2,\Omega}\,|||\boldsymbol{\theta}_I-\boldsymbol{\theta}_h|||_\Theta. \tag{66}$$

For the remaining terms, we can write

$$(\boldsymbol{\gamma},\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)-(\boldsymbol{\gamma}_h,P_h(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h))=(\boldsymbol{\gamma},(I-P_h)(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h))+(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,P_h(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)).$$

Let $P^0\boldsymbol{\gamma}$ be a piecewise constant approximation of $\boldsymbol{\gamma}$. Using (25), classical interpolation estimates, and (27) we easily deduce

$$\begin{aligned}(\boldsymbol{\gamma},(I-P_h)(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)) &=(\boldsymbol{\gamma}-P^0\boldsymbol{\gamma},(I-P_h)(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h))\\ &\le Ch|\boldsymbol{\gamma}|_{1,\Omega}\|(I-P_h)(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)\|_{0,\Omega}\\ &\le Ch|\boldsymbol{\gamma}|_{1,\Omega}|||\boldsymbol{\theta}_I-\boldsymbol{\theta}_h|||_\Theta.\end{aligned} \tag{67}$$

By subtracting (59) from the third equation of (30) we get

$$P_h(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)=\lambda^{-1}t^2(\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I)-\boldsymbol{\nabla}_h(w_h-w_I). \tag{68}$$

Thus, using this, the second equation of (45), and (46) we have:

$$\begin{aligned}(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,P_h(\boldsymbol{\theta}_I-\boldsymbol{\theta}_h)) &=\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I)-(\boldsymbol{\gamma}-\boldsymbol{\gamma}_h,\boldsymbol{\nabla}_h(w_h-w_I))\\ &=\lambda^{-1}t^2(\boldsymbol{\gamma}-\boldsymbol{\gamma}_I,\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I)-\lambda^{-1}t^2||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}^2\\ &+c_W(\boldsymbol{\gamma},w_h-w_I)\\ &\le\lambda^{-1}t^2||\boldsymbol{\gamma}-\boldsymbol{\gamma}_I||_{0,\Omega}||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}-\lambda^{-1}t^2||\boldsymbol{\gamma}_h-\boldsymbol{\gamma}_I||_{0,\Omega}^2\\ &+Ch|\boldsymbol{\gamma}|_{1,\Omega}|w_h-w_I|_{1,h}.\end{aligned} \tag{69}$$

We set, for the sake of brevity, $\delta\boldsymbol{\theta}=\boldsymbol{\theta}_I-\boldsymbol{\theta}_h$, $\delta\boldsymbol{\gamma}=\boldsymbol{\gamma}_I-\boldsymbol{\gamma}_h$. Substituting (52), (67), and (69) in (65), and using the interpolation estimates (55), (61) we obtain

$$
\begin{aligned}
\alpha\|\!|\delta\boldsymbol{\theta}|\!\|_\Theta^2 &+\lambda^{-1}t^2||\delta\boldsymbol{\gamma}||_{0,\Omega}^2 \le C(\|\!|\boldsymbol{\theta}-\boldsymbol{\theta}_I|\!\|_\Theta + h|\boldsymbol{\theta}|_{2,\Omega} + h|\boldsymbol{\gamma}|_{0,\Omega})\|\!|\delta\boldsymbol{\theta}|\!\|_\Theta \\
&+\lambda^{-1}t^2\|\boldsymbol{\gamma}-\boldsymbol{\gamma}_I\|_{0,\Omega}\|\delta\boldsymbol{\gamma}\|_{0,\Omega} + Ch|\boldsymbol{\gamma}|_{1,\Omega}|w_h - w_I|_{1,h} \\
&\le Ch(|\boldsymbol{\theta}|_{2,\Omega} + |\boldsymbol{\gamma}|_{0,\Omega})\|\!|\delta\boldsymbol{\theta}|\!\|_\Theta \\
&+Ch(\lambda^{-1}t^2|\boldsymbol{\gamma}|_{1,\Omega}\|\delta\boldsymbol{\gamma}\|_{0,\Omega} + |\boldsymbol{\gamma}|_{1,\Omega}|w_h - w_I|_{1,h}).
\end{aligned}
\tag{70}
$$

On the other hand, from (68) and (27) we have:

$$
|w_h - w_I|_{1,h} \le \lambda^{-1}t^2\|\delta\boldsymbol{\gamma}\|_{0,\Omega} + C\,\|\delta\boldsymbol{\theta}\|_{0,\Omega}. \tag{71}
$$

Using this and the inequality $2ab \le \varepsilon\, a^2 + b^2/\,\varepsilon$ in (70) we finally obtain:

$$
\|\!|\delta\boldsymbol{\theta}|\!\|_\Theta^2 + \lambda^{-1}t^2\|\delta\boldsymbol{\gamma}\|_{0,\Omega}^2 \le Ch^2(|\boldsymbol{\theta}|_{2,\Omega}^2 + \|\boldsymbol{\gamma}\|_{1,\Omega}^2 + \lambda^{-1}t^2|\boldsymbol{\gamma}|_{1,\Omega}^2), \tag{72}
$$

that is,

$$
\|\!|\boldsymbol{\theta}_I - \boldsymbol{\theta}_h|\!\|_\Theta + t\|\boldsymbol{\gamma}_I - \boldsymbol{\gamma}_h\|_\Gamma \le Ch(|\boldsymbol{\theta}|_{2,\Omega} + \|\boldsymbol{\gamma}\|_{1,\Omega}). \tag{73}
$$

We can finally conclude with the following convergence theorem.

Theorem 7 *Let $(\boldsymbol{\theta}, w, \boldsymbol{\gamma})$ be the solution of* (44)*, and let $(\boldsymbol{\theta}_h, w_h, \boldsymbol{\gamma}_h)$ that of* (30)*. Then, the following estimate holds:*

$$
\|\!|\boldsymbol{\theta} - \boldsymbol{\theta}_h|\!\|_\Theta + t\|\!|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h|\!\|_\Gamma \le Ch(|\boldsymbol{\theta}|_{2,\Omega} + \|\boldsymbol{\gamma}\|_{1,\Omega}), \tag{74}
$$

$$
\|\!|w - w_h|\!\|_W \le Ch(|\boldsymbol{\theta}|_{2,\Omega} + \|\boldsymbol{\gamma}\|_{1,\Omega} + |w|_{1,\Omega}), \tag{75}
$$

with C a positive constant independent of h.

Proof Estimate (74) follows from (73) and the triangle inequality. To derive (75), use first (71) and (73) to obtain

$$
|w_h - w_I|_{1,h} \le Ch(|\boldsymbol{\theta}|_{2,\Omega} + \|\boldsymbol{\gamma}\|_{1,\Omega}). \tag{76}
$$

Next, via duality argument and (51) it can be proved that, if Ω is convex, then (see, e.g., [2], [6])

$$
\|v\|_{0,\Omega} \le C|v|_{1,\Omega} \qquad \forall v \in W_h. \tag{77}
$$

The result (75) follows then by the triangle inequality. □

Remark 8 *The result of Theorem 7 is optimal with respect to the order of accuracy $O(h)$. It is not optimal, however, with respect to the regularity required on the solution. This regards, in particular, the regularity required for the shears $\boldsymbol{\gamma}$, that should be*

$$
t\|\boldsymbol{\gamma}\|_{1,\Omega} + \|\boldsymbol{\gamma}\|_{H(\mathrm{div};\Omega)}, \tag{78}
$$

instead of $\|\boldsymbol{\gamma}\|_{1,\Omega}$. An improved result, using only the regularity (78) *could possibly be obtained with arguments similar to those used in [7], but this goes beyond the aims of the present paper.*

5 Concluding remarks

As already pointed out, the choice of bubble spaces presented above is not, by far, the unique way in which the basic element (see below) can be made to work. We consider here as *basic element* the one in which each component of the rotations $\boldsymbol{\theta}_h$, as well as the transversal displacement w_h, are piecewise linear nonconforming functions, while the shears γ_h are piecewise constants.

Our analysis cannot be applied to this element: indeed, we remark that we cannot give up the d.o.f. (25) (Γ_h must contain at least the piecewise constants): but then we need, for defining $\boldsymbol{\theta}_I$, the degrees of freedom (54), otherwise we cannot prove (63), which is a crucial ingredient of Lemma 6, that, in turn, is the cornerstone of our proof strategy. In conclusion, **we need a bubble degree of freedom for $\boldsymbol{\theta}$**. Clearly, we are not obliged to use the nonconforming bubble (15). The cubic bubble ($\lambda_1\lambda_2\lambda_3$) would do the same job. Essentially, any element of the type

$$(P_1^{NC} \oplus B)^2 \quad \text{for } \boldsymbol{\theta}, \quad P_1^{NC} \quad \text{for } w, \quad (P_0)^2 \quad \text{for } \boldsymbol{\gamma},$$

(where $P_1^{NC} = P_1-$nonconforming, B =bubble) would work, with only very minor assumptions on the type of bubble (quadratic, cubic, pyramid, etc.).

We point out that **we do not need bubbles for w_h**, as far as we keep the shears to be piecewise constants. Indeed, for piecewise constant shears, we would only use the d.o.f.(25), and hence we only need property (63) in the crucial Lemma 6. But, for that, only the d.o.f (56) are necessary, and we are safe.

On the other hand we might want, as we did here, to have *the same degrees of freedom for $\boldsymbol{\theta}_h$ and w_h*. If we want this, then we can add a bubble to w_h as well. But **if we add a bubble to w_h we must also add its gradient in** Γ_h, or otherwise condition $\boldsymbol{\nabla}_h W_h \subset \Gamma_h$ (which is also crucial) will be violated.

Other changes can be done in the penalty terms, here defined as in (28). In the above discussion, we only used penalty for the $\boldsymbol{\theta}$ variable, and this is needed because otherwise the discrete Korn inequality would fail. We do not need penalty for w, but we might also use it, if we wish. We also point out that, in order to have the discrete Korn inequality, **we only need to penalize the jumps of the linear part of $\boldsymbol{\theta}$**. Hence, for instance, in our element we could substitute (28) with

$$\widetilde{p}_\Theta(\boldsymbol{\theta}, \boldsymbol{\eta}) := \sum_{e \in \mathcal{E}_h} \frac{1}{|e|} \int_e [Q_1\boldsymbol{\theta}] : [Q_1\boldsymbol{\eta}]\, dx, \tag{79}$$

where Q_1 is the L^2-projection on the space of polynomials of degree ≤ 1. This (that we actually **recommand**) will have the nice effect of allowing a simpler elimination of the bubbles (that, otherwise, will see each other at the interelement boundaries through the jump term: a most undesirable circumstance). This would not occur if we use true bubbles, instead of nonconforming ones.

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