DISCONTINUOUS GALERKIN APPROXIMATIONS FOR ELLIPTIC PROBLEMS

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Abstract. In this paper we analyze a discontinuous finite element method recently introduced by Bassi and Rebay for the approximation of elliptic problems. Stability and error estimates in various norms are proven.

1 Introduction

Discontinuous finite element methods are nowadays rather widely used, as a variant (or subclass) of finite volumes, in the numerical treatment of hyperbolic equations, in particular related to conservation laws: see, for instance, [11, 12, 17] and the references therein. Their use in the treatment of elliptic equations, in particular related to diffusion problems, is, at the same time, much older and much more recent. Indeed, the use of penalty methods to adapt C^0-elements to the discretization of fourth order problems goes back to the early seventies (see [14, 15]). For second order problems, the use of discontinuous elements with penalty can be traced back to a few years later, (see [2, 3],) but was afterwards rather abandoned. In recent times, as the use of discontinuous elements became fashionable for hyperbolic problems, there has been an attempt to use them as well in problems where small (but not negligible) diffusive terms are present together with major convective terms. The idea being, clearly, to use their good features for treating the convective terms, and "make them work" (one way or another) for treating the diffusive part as well. Among these attempts, we mention the Local Discontinuous Galerkin (LDG) method analyzed in [13] for time-dependent convection-diffusion systems, and the method proposed by Bassi and Rebay [4–7] for Navier-Stokes equations with high Reynolds number. The aim of this paper is to provide a solid mathematical background for the approach proposed by

Bassi and Rebay, and to trace properly its limits and its capabilities. In particular, we consider as a model problem the Laplace operator in a polygonal domain, in order to cast out the ability of the method to deal with diffusive terms. We rewrite the original formulation of [4–7] in a new and more elegant way, better suited for a mathematical investigation. We show that the original formulation can be rank-deficient when applied to stationary problems, and this, somehow, does not encourage its use in time dependent problems either. On the other hand, the variant proposed in [7] can be proven to be stable and optimal order accurate, at least with a minor modification in the choice of a parameter. Indeed, it is not yet clear whether for the value of the parameter proposed in [7] (simply equal to 1) the scheme is stable or not. What we prove here is that stability (together with all the other nice and desirable properties) is ensured if this value is strictly bigger than 3.

An outline of the paper is the following. In Section 2 we present the problem and the original formulation of [4–6]: using a more convenient notation we provide a new presentation which is better suited for theoretical investigation. Since, as already pointed out, the original formulation may present a potential instability, in Section 3 we discuss the stabilized version of [7]. Stability proofs and error estimates are then reported in Section 4. Finally, in Section 5 a different stabilization based on a penalty approach is introduced and investigated.

Acknowledgments. This research has been carried out within the project MIGALE, financed by ENEL Ricerca – Polo Termico (Pisa). We are also grateful, for the fruitful discussions we had with them, to F. Bassi (University of Ancona) and S. Rebay (University of Brescia) as well as to L. Brusa and S. Arnelli (from CISE) who, moreover, discovered experimentally the instability of the original formulation in the one-dimensional case.

2 Description of the Method

Let Ω be a two-dimensional convex polygonal domain, with boundary $\Gamma = \partial\Omega$ partitioned as $\Gamma = \overline{\Gamma}_N \cup \overline{\Gamma}_D$, with $\Gamma_N \cap \Gamma_D = \emptyset$. We consider the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ \dfrac{\partial u}{\partial n} = g_N & \text{on } \Gamma_N, \end{cases} \tag{1}$$

where f, g_D and g_N satisfy the usual regularity assumptions needed to write (1) in variational form (see, e.g., [18]). Introducing the auxiliary vector variable $\boldsymbol{\theta} = \nabla u$, problem (1) can be rewritten as

$$\begin{cases} \boldsymbol{\theta} - \nabla u = 0 & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{\theta} = f & \text{in } \Omega, \\ u = g_D & \text{on } \Gamma_D, \\ \boldsymbol{\theta} \cdot \boldsymbol{n} = g_N & \text{on } \Gamma_N, \end{cases} \tag{2}$$

where $\boldsymbol{n}$ is the outward normal unit vector to $\partial\Omega$. Let $\{\mathscr{T}_h\}_h$ be a regular family of triangulations of Ω, in the sense of [10]; we shall indicate by E the triangles of $\mathscr{T}_h$, and set $h = \max_{E\in\mathscr{T}_h} \operatorname{diam}(E)$. We denote by $\mathscr{E}_h$ the set of all edges of $\mathscr{T}_h$, by $\mathscr{E}_h'$ the set of the internal edges, by $\mathscr{E}_h^D$ the set of edges on Γ_D, and by $\mathscr{E}_h^N$ the set of edges on Γ_N. The length of the edge $e \in \mathscr{E}_h$ will be denoted by h_e.

The finite element spaces for the approximation of u and $\boldsymbol{\theta}$ will be denoted by V_h and $\boldsymbol{W}_h$ respectively. The precise definition of V_h and $\boldsymbol{W}_h$ will be given later; here we only assume that functions in V_h and $\boldsymbol{W}_h$ are (possibly) *discontinuous* along the edges, and no boundary values are enforced. We also require that

$$\nabla_h(V_h) \subset \boldsymbol{W}_h, \tag{3}$$

where ∇_h is the element by element gradient.

In [4–7] the following discrete variational formulation of (2) is considered:

$$
\begin{cases}
\text{find } (u_h,\, \boldsymbol{\theta}_h) \in V_h \times \boldsymbol{W}_h \text{ such that} \\
\displaystyle\sum_{E\in\mathscr{T}_h} \left[\int_E (\boldsymbol{\theta}_h - \nabla u_h)\cdot \boldsymbol{\tau}_h - \int_{\partial E} (u_h^0 - u_h)\, \boldsymbol{\tau}_h \cdot \boldsymbol{n} \right] = 0 \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \\
\displaystyle\sum_{E\in\mathscr{T}_h} \left[\int_E \boldsymbol{\theta}_h \cdot \nabla v_h - \int_{\partial E} \boldsymbol{\theta}_h^0 \cdot \boldsymbol{n}\, v_h \right] = \int_\Omega f v_h \quad \text{for all } v_h \in V_h,
\end{cases}
\tag{4}
$$

where u_h^0 and $\boldsymbol{\theta}_h^0$ are defined as follows. Let E be an element, and let e be an edge of E; let E^{ext} be the other element having e as an edge, and let u_h^{ext}, and $\boldsymbol{\theta}_h^{\text{ext}}$ denote the values of u_h, and $\boldsymbol{\theta}_h$, respectively, in E^{ext}. Then,

- $u_h^0 = \dfrac{u_h + u_h^{\text{ext}}}{2}$ on internal edges $e \in \mathscr{E}_h'$;
- $u_h^0 = g_D$ on Dirichlet boundary edges $e \in \mathscr{E}_h^D$;
- $u_h^0 = u_h$ on Neumann boundary edges $e \in \mathscr{E}_h^N$;

and

- $\boldsymbol{\theta}_h^0 = \dfrac{\boldsymbol{\theta}_h + \boldsymbol{\theta}_h^{\text{ext}}}{2}$ on internal edges $e \in \mathscr{E}_h$;
- $\boldsymbol{\theta}_h^0 = \boldsymbol{\theta}_h$ on Dirichlet boundary edges $e \in \mathscr{E}_h^D$;
- $\boldsymbol{\theta}_h^0 \cdot \boldsymbol{n} = g_N$ on Neumann boundary edges $e \in \mathscr{E}_h^N$.

Notice that the first equation in (4) corresponds to the condition $\boldsymbol{\theta} = \nabla u$, and the second one to the condition $-\operatorname{div} \boldsymbol{\theta} = f$.

Let us now introduce the *jump* of a function $v_h \in V_h$ on an edge e as

$$
[\![v_h]\!] =
\begin{cases}
v_h^+ \boldsymbol{n}^+ + v_h^- \boldsymbol{n}^- & \text{on internal edges,} \\
v_h \boldsymbol{n} & \text{on Dirichlet boundary edges,} \\
0 & \text{on Neumann boundary edges,}
\end{cases}
\tag{5}
$$

where $\boldsymbol{n}$ is the outward normal to the edge and the notation $(\cdot)^+$ and $(\cdot)^-$ indicates the value of the generic quantity $(\cdot)$ on the two elements sharing the same edge.

After some manipulations, equations (4) can be given the following (more elegant) form

$$
\begin{cases}
\displaystyle\sum_{E\in\mathscr{T}_h}\int_E(\boldsymbol{\theta}_h-\nabla u_h)\cdot\boldsymbol{\tau}_h+\sum_{e\in\mathscr{E}_h}\int_e [\![u_h]\!]\cdot\boldsymbol{\tau}_h^0=\sum_{e\in\mathscr{E}_h^D}\int_e g_D\,\boldsymbol{\tau}_h\cdot\boldsymbol{n}, & \text{(6a)}\\
\displaystyle\sum_{E\in\mathscr{T}_h}\int_E\boldsymbol{\theta}_h\cdot\nabla v_h-\sum_{e\in\mathscr{E}_h}\int_e\boldsymbol{\theta}_h^0\cdot[\![v_h]\!]=\sum_{e\in\mathscr{E}_h^N}\int_e g_N v_h+\int_\Omega f v_h. & \text{(6b)}
\end{cases}
$$

The system of equations (6) can be recognized as a saddle-point type problem. Actually, introducing the bilinear forms $a(\cdot,\cdot)$ on $\boldsymbol{W}_h\times\boldsymbol{W}_h$ and $b(\cdot,\cdot)$ on $V_h\times\boldsymbol{W}_h$ as

$$
a(\boldsymbol{\theta}_h,\boldsymbol{\tau}_h)=\int_\Omega\boldsymbol{\theta}_h\cdot\boldsymbol{\tau}_h, \tag{7}
$$

$$
b(u_h,\boldsymbol{\tau}_h)=-\sum_{E\in\mathscr{T}_h}\int_E\nabla u_h\cdot\boldsymbol{\tau}_h+\sum_{e\in\mathscr{E}_h}\int_e[\![u_h]\!]\cdot\boldsymbol{\tau}_h^0, \tag{8}
$$

problem (6) becomes

$$
\begin{cases}
\quad a(\boldsymbol{\theta}_h,\boldsymbol{\tau}_h)+b(u_h,\boldsymbol{\tau}_h)=\displaystyle\sum_{e\in\mathscr{E}_h^D}\int_e g_D\,\boldsymbol{\tau}_h\cdot\boldsymbol{n},\\
-b(v_h,\boldsymbol{\theta}_h)\qquad\qquad\qquad\;\;=\displaystyle\int_\Omega f v_h+\sum_{e\in\mathscr{E}_h^N}\int_e g_N v_h.
\end{cases} \tag{9}
$$

It is then clear that an *Inf-Sup* condition should be satisfied in order to ensure existence and uniqueness of the solution of (9). See, e.g., [8]. We shall discuss this point in the next section.

Now, we can derive a single variational equation by solving equation (6) for $\boldsymbol{\theta}_h$. To this end, we first define the space $\widetilde{V}$ as

$$
\widetilde{V}=\prod_{E\in\mathscr{T}_h}H^1(E), \tag{10}
$$

and the affine operator $\boldsymbol{R}_{g_D}:\widetilde{V}\to\boldsymbol{W}_h$ in the following way. Given $w\in\widetilde{V}$, we define $\boldsymbol{R}_{g_D}(w)\in\boldsymbol{W}_h$ as the solution of the variational problem

$$
\int_\Omega\boldsymbol{R}_{g_D}(w)\cdot\boldsymbol{\tau}_h=-\sum_{e\in\mathscr{E}_h}\int_e[\![w]\!]\cdot\boldsymbol{\tau}_h^0+\sum_{e\in\mathscr{E}_h^D}\int_e g_D\,\boldsymbol{\tau}_h\cdot\boldsymbol{n}\quad\text{for all }\boldsymbol{\tau}_h\in\boldsymbol{W}_h. \tag{11}
$$

From (6) and (3) we have then

$$\boldsymbol{R}_{g_D}(u_h) = \boldsymbol{\theta}_h - \nabla u_h. \tag{12}$$

We also define the linear operator $\boldsymbol{R} = \boldsymbol{R}_0 : \widetilde{V} \to \boldsymbol{W}_h$ by

$$\int_\Omega \boldsymbol{R}(w) \cdot \boldsymbol{\tau}_h = -\sum_{e \in \mathscr{E}_h} \int_e [\![w]\!] \cdot \boldsymbol{\tau}_h^0 \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{W}_h, \tag{13}$$

and we notice that the following relationship holds:

$$\int_\Omega \boldsymbol{R}_{g_D}(w) \cdot \boldsymbol{\tau}_h = \int_\Omega \boldsymbol{R}(w) \cdot \boldsymbol{\tau}_h + \sum_{e \in \mathscr{E}_h^D} \int_e g_D \, \boldsymbol{\tau}_h \cdot \boldsymbol{n}. \tag{14}$$

Using (13) (with $\boldsymbol{\tau}_h = \boldsymbol{\theta}_h$ and $w = v_h$), and (12) in equation (6), the scheme becomes:

$$\sum_{E \in \mathscr{T}_h} \int_E \left[\nabla u_h + \boldsymbol{R}_{g_D}(u_h)\right] \cdot \left[\nabla v_h + \boldsymbol{R}(v_h)\right] = \sum_{e \in \mathscr{E}_h^N} \int_e g_N v_h + \int_\Omega f v_h, \tag{15}$$

or equivalently, using (14),

$$\begin{aligned} \sum_{E \in \mathscr{T}_h} \int_E \left[\nabla u_h + \boldsymbol{R}(u_h)\right] \cdot \left[\nabla v_h + \boldsymbol{R}(v_h)\right] = \sum_{e \in \mathscr{E}_h^N} \int_e g_N v_h + \int_\Omega f v_h \\ - \sum_{e \in \mathscr{E}_h^D} \int_e g_D \left[\nabla v_h + \boldsymbol{R}(v_h)\right] \cdot \boldsymbol{n}. \end{aligned} \tag{16}$$

3 The $\mathbb{P}_k$-$\mathbb{P}_k$ approximation

We are interested in the study of system (9) for discontinuous piecewise polynomials of degree $k \geq 1$, for both $v_h \in V_h$ and $\boldsymbol{\tau}_h \in \boldsymbol{W}_h$. Hence, we define, for $k \geq 1$,

$$V_h = \left\{ v_h \in L^2(\Omega) \text{ such that } v_{h|E} \in \mathbb{P}_k(E) \text{ for all } E \in \mathscr{T}_h \right\}, \tag{17}$$

$$\boldsymbol{W}_h = \left\{ \boldsymbol{\tau}_h \in \left[L^2(\Omega)\right]^2 \text{ such that } \boldsymbol{\tau}_{h|E} \in \left[\mathbb{P}_k(E)\right]^2 \text{ for all } E \in \mathscr{T}_h \right\}. \tag{18}$$

As already mentioned, an *Inf-Sup* condition relating the spaces V_h and $\boldsymbol{W}_h$ is needed in order to guarantee the nonsingularity of the matrix associated with (9). Unfortunately, the *Inf-Sup* condition does not hold for this choice of spaces, as shown in [9] through a counter-example. Since an a priori control over the u_h variable cannot be provided, it seems quite natural to add to the second equation of (9) a suitable stabilizing term. We shall describe here the modification proposed in [7]. Mimicking the definition of $\boldsymbol{R}_{g_D}$, for each Dirichlet boundary edge e of the triangulation we define the affine operator $\boldsymbol{r}_{e,g_D} : \widetilde{V} \to \boldsymbol{W}_h$ by

$$\int_\Omega \boldsymbol{r}_{e,g_D}(w) \cdot \boldsymbol{\tau}_h = -\int_e w\, \boldsymbol{\tau}_h \cdot \boldsymbol{n} + \int_e g_D \boldsymbol{\tau}_h \cdot \boldsymbol{n} \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{W}_h. \tag{19}$$

If e is an internal edge, $\boldsymbol{r}_{e,g_D}(w)$ is defined by

$$\int_\Omega \boldsymbol{r}_{e,g_D}(w) \cdot \boldsymbol{\tau}_h = -\int_e [\![w]\!] \cdot \boldsymbol{\tau}_h^0 \quad \text{for all } \boldsymbol{\tau}_h \in \boldsymbol{W}_h \tag{20}$$

(note that in this case $\boldsymbol{r}_{e,g_D}$ does not depend on g_D), and for Neumann boundary edges $\boldsymbol{r}_{e,g_D}$ is defined to be zero. Finally, as before we set $\boldsymbol{r}_e = \boldsymbol{r}_{e,0}$. It can be easily seen that the following relationship between $\boldsymbol{R}_{g_D}$ and $\boldsymbol{r}_{e,g_D}$ holds: for any triangle $E \in \mathscr{T}_h$, we have

$$\sum_{e \subset \partial E} \boldsymbol{r}_{e,g_D} = \boldsymbol{R}_{g_D} \quad \text{on } E. \tag{21}$$

If e is an internal edge, it is clear from definition (20) that the support of $\boldsymbol{r}_{e,g_D}$ is contained in the union of the two triangles sharing the edge e. The modification proposed by Bassi and Rebay consists in *replacing* in (15), for each $E \in \mathscr{T}_h$,

$$\text{the term} \quad \int_E \boldsymbol{R}_{g_D}(u_h) \cdot \boldsymbol{R}(v_h) \quad \text{by} \quad \sum_{e \subset \partial E} \int_\Omega \boldsymbol{r}_{e,g_D}(u_h) \cdot \boldsymbol{r}_e(v_h). \tag{22}$$

This procedure can be interpreted in the following way. As we shall see more precisely in the next Section, the quantity $\boldsymbol{r}_e(v_h)$ allows to control the jump of v_h on e (see Lemma 2); hence, a natural stabilization of (9) consists in *adding* to the left-hand side of the second equation the term

$$s \sum_{e \in \mathscr{E}_h} \int_\Omega \boldsymbol{r}_{e,g_D}(u_h) \cdot \boldsymbol{r}_e(v_h), \tag{23}$$

where $s > 0$ is a parameter (to be chosen later on), thus obtaining the new scheme

$$\begin{aligned} &\sum_{E \in \mathscr{T}_h} \int_E \left[\nabla u_h + \boldsymbol{R}_{g_D}(u_h)\right] \cdot \left[\nabla v_h + \boldsymbol{R}(v_h)\right] \\ &\qquad + s \sum_{e \in \mathscr{E}_h} \int_\Omega \boldsymbol{r}_{e,g_D}(u_h) \cdot \boldsymbol{r}_e(v_h) = \sum_{e \in \mathscr{E}_h^N} \int_e g_N v_h + \int_\Omega f v_h. \end{aligned} \tag{24}$$

Indeed, as we also point out at the end of Section 4, this stabilization works for every positive s. In the next Section, we shall show that if s is large enough ($s > 3$), we can also supress the term $\int_E \boldsymbol{R}_{g_D}(u_h) \cdot \boldsymbol{R}(v_h)$ in (24), obtaining the following scheme, equivalent to the formulation of Bassi and Rebay [7] (when $s = 1$):

$$\sum_{E \in \mathscr{T}_h} \int_E \left[\nabla u_h \cdot \nabla v_h + \nabla u_h \cdot \boldsymbol{R}(v_h) + \boldsymbol{R}_{g_D}(u_h) \cdot \nabla v_h \right] + s \sum_{e \in \mathscr{E}_h} \int_\Omega \boldsymbol{r}_{e,g_D}(u_h) \cdot \boldsymbol{r}_e(v_h) = \sum_{e \in \mathscr{E}_h^N} \int_e g_N v_h + \int_\Omega f v_h. \quad (25)$$

The advantage of this scheme with respect to (24) is that the stiffness matrix of (25) is much more sparse than that of (24). Indeed, if we take a v_h having support inside one element E (far from the boundary), we see that the elements involved in (24) are always 10, while the elements involved in (25) are always 4, as depicted in Figure 1.

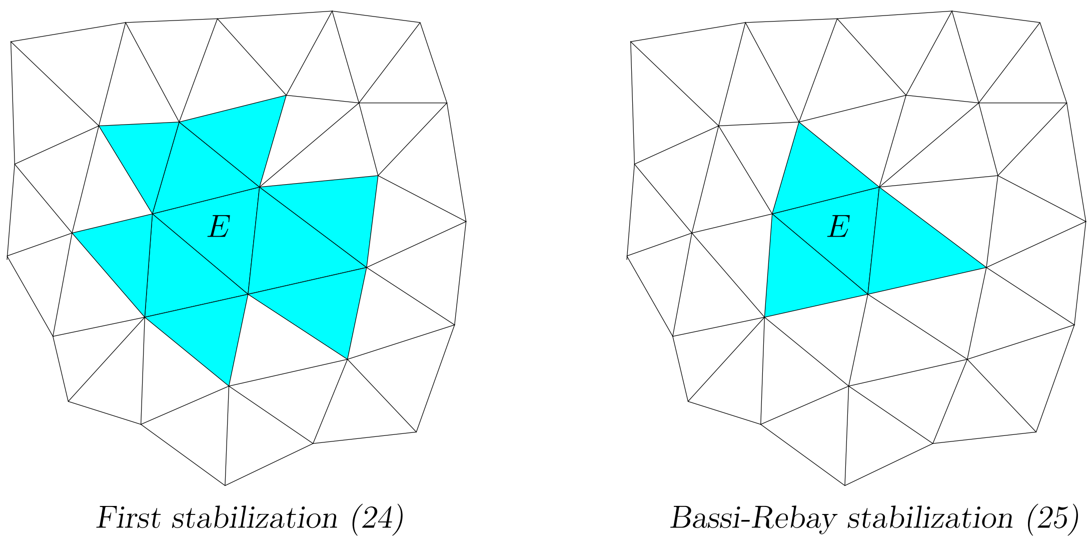

Figure 1: Sparsity of the stiffness matrix

For the sake of simplicity, in this paper we shall limit ourselves to study in detail the case of Dirichlet homogeneous boundary conditions. Hence, from now on we will assume $\Gamma_N = \emptyset$, $\Gamma_D = \partial\Omega$, and $g_D = 0$, so that the continuous problem reads

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (26)$$

Accordingly, we define the bilinear form $a_h(\cdot,\cdot)$ after (25) as

$$a_h(u_h, v_h) = \sum_{E\in\mathscr{T}_h} \int_E \left[\nabla u_h \cdot \nabla v_h + \nabla u_h \cdot \boldsymbol{R}(v_h) + \boldsymbol{R}(u_h) \cdot \nabla v_h\right] \\ + s \sum_{e\in\mathscr{E}_h} \int_\Omega \boldsymbol{r}_e(u_h) \cdot \boldsymbol{r}_e(v_h), \quad (27)$$

and the discrete variational formulation becomes

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) = (f, v_h), \quad \text{for all } v_h \in V_h. \end{cases} \quad (28)$$

Error estimates will be obtained in the following mesh-dependent norm:

$$|||v|||^2 = \sum_{E\in\mathscr{T}_h} |v|^2_{1,E} + \sum_{e\in\mathscr{E}_h} \|\boldsymbol{r}_e(v)\|^2_{0,\Omega}, \quad v \in \widetilde{V}, \quad (29)$$

and, as usual, $\|\cdot\|_{m,S}$, $|\cdot|_{m,S}$ denote the norm and seminorm, respectively, in the Sobolev space $H^m(S)$. Notice that $|||\cdot|||$ is only a semi-norm in $\widetilde{V}$, while it is a norm in $V_h + H^1_0(\Omega)$.

Remark. Using definitions (13)-(14) in (25) the bilinear form can be rewritten equivalently as

$$a_h(u_h, v_h) = \sum_{E\in\mathscr{T}_h} \int_E \nabla u_h \cdot \nabla v_h - \sum_{e\in\mathscr{E}_h} \int_e [\![u_h]\!] \cdot (\nabla v_h)^0 - \sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla u_h)^0 \\ + s \sum_{e\in\mathscr{E}_h} \int_\Omega \boldsymbol{r}_{e,g_D}(u_h) \cdot \boldsymbol{r}_e(v_h). \quad (30)$$

One can then recognize similarities with other formulations studied, for instance, in [2, 15, 19], the only difference being in the last term of (30).

4 Error Estimates

We shall prove in this section coercivity and continuity on V_h of the bilinear form $a_h(\cdot,\cdot)$ with respect to the norm $|||\cdot|||$ defined in (29). Moreover, under the regularity assumption $u \in H^{k+1}(\Omega) \cap H^1_0(\Omega)$, with $k \geq 1$, we shall prove the following error estimates:

- $\|u - u_h\| \le C\, h^k |u|_{k+1,\Omega}$,
- $\|u - u_h\|_{0,\Omega} \le C\, h^{k+1} |u|_{k+1,\Omega}$,

where u is the solution of (26), and u_h is the solution of (28). Here and in the sequel, C will denote a generic constant depending only on the minimum angle of the triangulation.

Proposition 1 *There exists a constant $M > 0$, independent of h, such that*

(i) $a_h(u_h, v_h) \le M \|u_h\| \, \|v_h\|$, *for all* $u_h, v_h \in V_h$.

Moreover, if $s > 3$, there exists a constant $\alpha > 0$, independent of h, such that

(ii) $a_h(v_h, v_h) \ge \alpha \|v_h\|^2$, *for all* $v_h \in V_h$.

Proof. We first observe that an easy consequence of (21) is

$$\|\boldsymbol{R}(v_h)\|_{0,E}^2 \le 3 \sum_{e \subset \partial E} \|\boldsymbol{r}_e(v_h)\|_{0,E}^2. \tag{31}$$

Then, *(i)* follows immediately from the definition of $a_h(u_h, v_h)$ and inequality (31). In order to prove *(ii)*, notice that, since the support of each $\boldsymbol{r}_e$ is the union of the triangles sharing the edge e, we can write

$$\sum_{e \in \mathscr{E}_h} \|\boldsymbol{r}_e(v_h)\|_{0,\Omega}^2 = \sum_{E \in \mathscr{T}_h} \sum_{e \subset \partial E} \|\boldsymbol{r}_e(v_h)\|_{0,E}^2. \tag{32}$$

Using (32), inequality $2ab \le \varepsilon a^2 + \dfrac{1}{\varepsilon} b^2$, and (31) we deduce

$$\begin{aligned}
a_h(v_h, v_h) &= \sum_{E \in \mathscr{T}_h} \left[|v_h|_{1,E}^2 + 2 \int_E \boldsymbol{R}(v_h) \cdot \nabla v_h + s \sum_{e \subset \partial E} \|\boldsymbol{r}_e(v_h)\|_{0,E}^2 \right] \\
&\ge \sum_{E \in \mathscr{T}_h} \left[(1-\varepsilon) |v_h|_{1,E}^2 - \frac{1}{\varepsilon} \|\boldsymbol{R}(v_h)\|_{0,E}^2 + s \sum_{e \subset \partial E} \|\boldsymbol{r}_e(v_h)\|_{0,E}^2 \right] \\
&\ge \sum_{E \in \mathscr{T}_h} \left[(1-\varepsilon) |v_h|_{1,E}^2 + \left(s - \frac{3}{\varepsilon} \right) \sum_{e \subset \partial E} \|\boldsymbol{r}_e(v_h)\|_{0,E}^2 \right].
\end{aligned}$$

Then, *(ii)* holds with $\alpha = \min\left(1 - \varepsilon, s - \dfrac{3}{\varepsilon}\right)$, which is positive whenever $\dfrac{3}{s} < \varepsilon < 1$. ■

The following properties will be used in the sequel.

Lemma 1 *Let $\varphi \in H^1(\Omega)$, with $\Delta\varphi \in L^2(\Omega)$, and let $v_h \in V_h$. Then*

$$\sum_{E\in\mathscr{T}_h} \int_{\partial E} v_h \frac{\partial \varphi}{\partial n} = \sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla\varphi)^0. \tag{33}$$

Proof. The regularity of φ implies that $\nabla\varphi \cdot \boldsymbol{n}$ is continuous across the interelement boundaries, that is, $\boldsymbol{n}^+ \cdot \nabla\varphi^+ + \boldsymbol{n}^- \cdot \nabla\varphi^- = 0$. Thus, recalling (5), we have

$$\begin{aligned}
\sum_{E\in\mathscr{T}_h} \int_{\partial E} v_h \frac{\partial \varphi}{\partial n} &= \sum_{e\in\mathscr{E}_h'} \int_e \left[v_h^+ \boldsymbol{n}^+ \cdot \nabla\varphi^+ + v_h^- \boldsymbol{n}^- \cdot \nabla\varphi^- \right] + \sum_{e\in\mathscr{E}_h^D} \int_e v_h \boldsymbol{n} \cdot \nabla\varphi \\
&= \sum_{e\in\mathscr{E}_h'} \int_e \left[v_h^+ \boldsymbol{n}^+ \cdot \frac{\nabla\varphi^+ + \nabla\varphi^-}{2} + v_h^- \boldsymbol{n}^- \cdot \frac{\nabla\varphi^+ + \nabla\varphi^-}{2} \right] + \sum_{e\in\mathscr{E}_h^D} \int_e v_h \boldsymbol{n} \cdot \nabla\varphi \\
&= \sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla\varphi)^0. \qquad \blacksquare
\end{aligned}$$

Lemma 2 *There exist two positive constants C_1 and C_2, independent of h, such that*

(i) $\|[\![v_h]\!]\|_{0,e} \le C_1 h_e^{1/2} \|\boldsymbol{r}_e(v_h)\|_{0,\Omega}$

(ii) $\|\boldsymbol{r}_e(v_h)\|_{0,\Omega} \le C_2 h_e^{-1/2} \|[\![v_h]\!]\|_{0,e}$

for each $v_h \in V_h$ and for each $e \in \mathscr{E}_h$.

Proof. Let e be an edge of E. Given $\varphi_h \in \mathbb{P}_k(e)$, let $P_e(\varphi_h) \in \mathbb{P}_k(E)$ be the extension of φ_h on E, which is constant along the lines orthogonal to the edge e. It follows immediately that

$$\|P_e(\varphi_h)\|_{0,E} \le C h_e^{1/2} \|\varphi_h\|_{0,e}. \tag{34}$$

For a vector $\boldsymbol{\varphi}_h \in [\mathbb{P}_k(e)]^2$, $P_e(\boldsymbol{\varphi}_h)$ is defined component-wise. Taking in (19) $w = v_h$, and $\boldsymbol{\tau}_h \in \boldsymbol{W}_h$ defined by $\boldsymbol{\tau}_{h|E} = P_e([\![v_h]\!])$, and $\boldsymbol{\tau}_h = 0$ elsewhere, we have

$$\frac{1}{2}\|[\![v_h]\!]\|_{0,e}^2 \le \int_E |\boldsymbol{r}_e(v_h) \cdot P_e([\![v_h]\!])| \le \|\boldsymbol{r}_e(v_h)\|_{0,E} \, \|P_e([\![v_h]\!])\|_{0,E} \, . \tag{35}$$

Then, inequality *(i)* follows by using (34) in (35) and summing over all $E \in \mathscr{T}_h$. In order to prove *(ii)*, we take $\boldsymbol{\tau}_h = \boldsymbol{r}_e(v_h)$ in (20). Thus, we find

$$\|\boldsymbol{r}_e(v_h)\|_{0,\Omega}^2 \le \|[\![v_h]\!]\|_{0,e} \, \|(\boldsymbol{r}_e(v_h))^0\|_{0,e} \le C \, h_e^{-1/2} \, \|[\![v_h]\!]\|_{0,e} \, \|\boldsymbol{r}_e(v_h)\|_{0,\Omega}, \tag{36}$$

where the last inequality follows by a simple scaling argument, see [16]. $\blacksquare$

Lemma 3 *Let $\varphi \in \prod_{E\in\mathscr{T}_h} H^2(E)$ and $v_h \in V_h$. Then*

$$\sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla\varphi)^0 \le C \sum_{E\in\mathscr{T}_h} \sum_{e\subset\partial E} \|\boldsymbol{r}_e(v_h)\|_{0,\Omega} \left(|\varphi|_{1,E} + h_e|\varphi|_{2,E}\right). \tag{37}$$

Proof. The trace inequality (see [1, 2]) gives

$$\|\nabla\varphi\|_{0,e}^2 \le C\, \left(h_e^{-1}|\varphi|_{1,E}^2 + h_e|\varphi|_{2,E}^2\right), \tag{38}$$

where the constant C depends only on the minimum angle bound. Using (38) and Lemma 2 *(i)* we obtain

$$\int_e [\![v_h]\!] \cdot \nabla\varphi \le \; \|[\![v_h]\!]\|_{0,e}\, \|\nabla\varphi\|_{0,e} \le \; C\|\boldsymbol{r}_e(v_h)\|_{0,\Omega} \left(|\varphi|_{1,E} + h_e|\varphi|_{2,E}\right), \tag{39}$$

and the thesis follows since

$$\sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla\varphi)^0 = \frac{1}{2} \sum_{E\in\mathscr{T}_h} \sum_{e\subset\partial E} \int_e [\![v_h]\!] \cdot \nabla\varphi. \qquad \blacksquare$$

We are now able to prove our convergence theorem.

Theorem 1 *Let u and u_h be the solutions of (26) and (28), respectively. Then, the following estimate holds:*

$$|\!|\!| u - u_h |\!|\!| \le C h^k |u|_{k+1,\Omega}. \tag{40}$$

Proof. Let $\widetilde{V}_h$ be the usual conforming finite element space

$$\widetilde{V}_h = \{v_h \in H_0^1(\Omega) \text{ such that } v_{h|E} \in \mathbb{P}_k(E)\}, \quad \text{with } k \ge 1, \tag{41}$$

and let $u_I \in \widetilde{V}_h$ be the $\mathbb{P}_k$-interpolant of u. The definition of $|\!|\!| \cdot |\!|\!|$ yields

$$|\!|\!| u - u_I |\!|\!|^2 = \sum_{E\in\mathscr{T}_h} |u - u_I|_{1,E}^2 \le C\, h^{2k}\, |u|_{k+1,\Omega}^2. \tag{42}$$

From Proposition 1 we have

$$\begin{aligned} \alpha |\!|\!| u_I - u_h |\!|\!|^2 &\le a_h(u_I - u_h, u_I - u_h) \\ &= a_h(u_I - u, u_I - u_h) + a_h(u - u_h, u_I - u_h) \\ &\le M |\!|\!| u_I - u |\!|\!| \, |\!|\!| u_I - u_h |\!|\!| + a_h(u - u_h, u_I - u_h). \end{aligned} \tag{43}$$

It is now easy to see that

$$a_h(u - u_h, u_I - u_h) = 0. \tag{44}$$

Indeed, using definition (30) for $a_h(\cdot\,,\cdot\,)$ and integration by parts we obtain

$$\begin{aligned} a_h(u - u_h, u_I - u_h) &= a_h(u, u_I - u_h) - a_h(u_h, u_I - u_h) \\ &= \sum_{E\in\mathscr{T}_h} \int_E \nabla u \cdot \nabla(u_I - u_h) - \sum_{e\in\mathscr{E}_h} \int_e (\nabla u)^0 \cdot [\![u_I - u_h]\!] - \sum_{E\in\mathscr{T}_h} \int_E f\ (u_I - u_h) \quad (45) \\ &= \sum_{E\in\mathscr{T}_h} \left[\int_E (-\Delta u - f)(u_I - u_h) + \int_{\partial E} \frac{\partial u}{\partial n}(u_I - u_h) \right] - \sum_{e\in\mathscr{E}_h} \int_e (\nabla u)^0 \cdot [\![u_I - u_h]\!], \end{aligned}$$

and (44) follows from Lemma 1 . Then, (40) is a consequence of (42), (43), and the triangle inequality. ■

We shall now prove an L^2-estimate.

Theorem 2 *Let u and u_h be the solutions of (26) and (28), respectively. Then, the following estimate holds:*

$$\|u - u_h\|_{0,\Omega} \le C h^{k+1} |u|_{k+1,\Omega}. \tag{46}$$

Proof. As standard in duality arguments, we consider the following auxiliary problem

$$\begin{cases} -\Delta w = u - u_h & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{47}$$

for which the regularity estimate

$$\|w\|_{2,\Omega} \le C\,\|u - u_h\|_{0,\Omega} \tag{48}$$

holds. We denote by $\widetilde{w}_h \in \widetilde{V}_h$ the solution of the conforming finite element problem

$$\int_\Omega \nabla\widetilde{w}_h \cdot \nabla\widetilde{v}_h = \int_\Omega (u - u_h)\,\widetilde{v}_h \quad \text{for all } \widetilde{v}_h \in \widetilde{V}_h, \tag{49}$$

and we recall that (see e.g. [10])

$$\|w - \widetilde{w}_h\|_{1,\Omega} \le C\,h |w|_{2,\Omega}. \tag{50}$$

From equation (47), integrating by parts and keeping the interelement terms, since u_h is discontinuous, we obtain

$$\|u-u_h\|_{0,\Omega}^2 = \sum_{E\in\mathscr{T}_h}\int_E -\Delta w(u-u_h) = \sum_{E\in\mathscr{T}_h}\left[\int_E \nabla w\cdot\nabla(u-u_h) - \int_{\partial E}\frac{\partial w}{\partial n}(u-u_h)\right]$$
$$= \sum_{E\in\mathscr{T}_h}\left[\int_E \nabla(w-\widetilde{w}_h)\cdot\nabla(u-u_h) + \int_E \nabla\widetilde{w}_h\cdot\nabla(u-u_h) + \int_{\partial E}\frac{\partial w}{\partial n}\,(u_h-u)\right]. \quad (51)$$

Since u_h is the solution of (28), and u that of (26), using definition (30) we have

$$\sum_{E\in\mathscr{T}_h}\int_E \nabla(u-u_h)\cdot\nabla\widetilde{w}_h = -\sum_{e\in\mathscr{E}_h}\int_e [\![u_h]\!]\cdot(\nabla\widetilde{w}_h)^0. \quad (52)$$

Therefore, (51) reduces to

$$\begin{aligned}\|u-u_h\|_{0,\Omega}^2 = &\sum_{E\in\mathscr{T}_h}\left[\int_E \nabla(w-\widetilde{w}_h)\cdot\nabla(u-u_h) + \int_{\partial E}\frac{\partial w}{\partial n}\,(u_h-u)\right]\\ &-\sum_{e\in\mathscr{E}_h}\int_e [\![u_h]\!]\cdot(\nabla\widetilde{w}_h)^0.\end{aligned} \quad (53)$$

By Lemma 1 and Lemma 3 we have

$$-\sum_{e\in\mathscr{E}_h}\int_e [\![u_h]\!]\cdot(\nabla\widetilde{w}_h)^0 + \sum_{e\in\mathscr{E}_h}\int_e [\![u_h-u]\!]\cdot(\nabla w)^0 = \sum_{e\in\mathscr{E}_h}\int_e [\![u_h]\!]\cdot[\nabla(w-\widetilde{w}_h)]^0 \leq$$
$$C\sum_{E\in\mathscr{T}_h}\sum_{e\subset\partial E}\|\boldsymbol{r}_e(u_h)\|_{0,\Omega}\left(|w-\widetilde{w}_h|_{1,E} + h_e|w-\widetilde{w}_h|_{2,E}\right). \quad (54)$$

Thus, substituting (54) in (53), and using (50) we deduce

$$\|u-u_h\|_{0,\Omega}^2 \leq C\,h|\!|\!|u-u_h|\!|\!|\,|w|_{2,\Omega}. \quad (55)$$

Finally, applying estimates (48) and (40) we conclude the proof. ■

Remark. A quick glance at the proofs of Proposition 1 and Theorems 1 and 2 immediately shows that the same results, namely error estimates (40) and (46), can be obtained for the scheme (24) for every $s > 0$.

5 A penalty method

In this section we present and analyze a variant of the scheme introduced by Bassi and Rebay. This variant presents some computational advantages, as it reduces the number of integrals to be computed when building the elementary matrices. On the other hand, with this variant the scheme becomes a penalty method, and very large coefficients might be introduced in the matrix, mainly when high-order polynomials are used. Let us consider the problem:

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, v_h) = (f, v_h), \quad \text{for all } v_h \in V_h \end{cases} \tag{56}$$

with $a_h(\cdot,\cdot)$ now defined by:

$$a_h(u_h, v_h) = \sum_{E\in\mathscr{T}_h} \int_E \nabla u_h \cdot \nabla v_h + \sum_{e\in\mathscr{E}_h} s(h_e) \int_\Omega \boldsymbol{r}_e(u_h) \cdot \boldsymbol{r}_e(v_h). \tag{57}$$

In (57) $s(h_e)$ is a positive function which tends to $+\infty$ when h_e tends to zero. For reasons which will become clear in the proof, we choose

$$s(h_e) = \frac{1}{h_e^{2k}}, \tag{58}$$

k being the order of the polynomials used in the approximation. We define the norm

$$\|\!|v|\!\|_h^2 = a_h(v, v) = \sum_{E\in\mathscr{T}_h} |v|_{1,E}^2 + \sum_{e\in\mathscr{E}_h} s(h_e)\|\boldsymbol{r}_e(v)\|_{0,\Omega}^2, \;\; v \in \widetilde{V}, \tag{59}$$

and, always with the same regularity assumptions of the previous section, we shall prove the following error estimates:

- $\|\!|u - u_h|\!\|_h \le C\, h^k |u|_{k+1,\Omega}$,
- $\|u - u_h\|_{0,\Omega} \le C\, h^{k+1} |u|_{k+1,\Omega}$,

where u is the solution of (26), and u_h that of (56). Let us notice that the continuity and the coercivity of $a_h(\cdot,\cdot)$, with respect to the norm $\|\!|\cdot|\!\|_h$, are now straightforward (with $M = \alpha = 1$). The following modification of Lemma 3 will also be useful :

Lemma 4 *Let be $\varphi \in \prod_{E\in\mathscr{T}_h} H^2(E)$ and $v_h \in V_h$. Then*

$$\sum_{e\in\mathscr{E}_h} \int_e [\![v_h]\!] \cdot (\nabla\varphi)^0 \le C\, \|\!|v_h|\!\|_h \left(\sum_{E\in\mathscr{T}_h} \sum_{e\subset\partial E} s(h_e)^{-1} \|\varphi\|_{2,E}^2 \right)^{1/2}. \tag{60}$$

Proof. From Lemma 3 we easily obtain

$$\sum_{e\in\mathscr{E}_h}\int_e [\![v_h]\!]\cdot(\nabla\varphi)^0 \le C\sum_{E\in\mathscr{T}_h}\sum_{e\subset\partial E}\|\boldsymbol{r}_e(v_h)\|_{0,\Omega}\|\varphi\|_{2,E}s(h_e)^{-1/2}s(h_e)^{1/2}. \tag{61}$$

The thesis follows immediately by applying Cauchy-Schwarz inequality and (32). ■

We can now prove the convergence theorem.

Theorem 3 *Let u and u_h be the solutions of (26) and (28), respectively. Then, the following estimate holds:*

$$|\!|\!| u-u_h |\!|\!|_h \le Ch^k|u|_{k+1,\Omega}. \tag{62}$$

Proof. Proceeding as in Theorem 1, let again $u_I\in\widetilde{V}_h$ be the $\mathbb{P}_k$–interpolant of u. The definition of $|\!|\!|\cdot|\!|\!|_h$ yields

$$|\!|\!| u-u_I |\!|\!|_h^2 = \sum_{E\in\mathscr{T}_h}|u-u_I|^2_{1,E}\le C\,h^{2k}\,|u|^2_{k+1,\Omega}. \tag{63}$$

From definition (59) it easily follows that

$$\begin{aligned}|\!|\!| u_I-u_h |\!|\!|_h^2 &\le a_h(u_I-u_h,u_I-u_h)\\ &= a_h(u_I-u,u_I-u_h)+a_h(u-u_h,u_I-u_h)\\ &\le |\!|\!| u_I-u |\!|\!|_h\,|\!|\!| u_I-u_h |\!|\!|_h + a_h(u-u_h,u_I-u_h).\end{aligned} \tag{64}$$

Using (57) and (56), integrating by parts, and applying Lemma 1 we have that

$$\begin{aligned}a_h(u-u_h,u_I-u_h) &= a_h(u,u_I-u_h)-a_h(u_h,u_I-u_h)\\ &= \sum_{E\in\mathscr{T}_h}\left[\int_E\nabla u\cdot\nabla(u_I-u_h)-\int_E f\ (u_I-u_h)\right]\\ &= \sum_{E\in\mathscr{T}_h}\left[\int_E(-\Delta u-f)(u_I-u_h)+\int_{\partial E}\frac{\partial u}{\partial n}(u_I-u_h)\right]\\ &= \sum_{e\in\mathscr{E}_h}\int_e[\![u_I-u_h]\!]\cdot(\nabla u)^0.\end{aligned} \tag{65}$$

From Lemma 4 we have

$$\left|\sum_{e\in\mathscr{E}_h}\int_e[\![u_I-u_h]\!]\cdot(\nabla u)^0\right| \le C\,|\!|\!| u_I-u_h |\!|\!|_h\left(\sum_{E\in\mathscr{T}_h}\sum_{e\subset\partial E}s(h_e)^{-1}\|u\|^2_{2,E}\right)^{1/2}, \tag{66}$$

and inserting inequality (66) in (65) and (64), we obtain

$$\|u_I - u_h\|_h^2 \leq \|u_I - u_h\|_h \, \|u_I - u\|_h + C \, \|u_I - u_h\|_h \left(\sum_{E\in\mathscr{T}_h} \sum_{e\subset\partial E} s(h_e)^{-1} \|u\|_{2,E}^2 \right)^{1/2} . \quad (67)$$

The result follows then from (63), (67), and triangle inequality, by choosing $s(h_e)$ as in (58). ■

A duality argument allows us to prove the following L^2-estimate.

Theorem 4 *Let u and u_h be the solutions of (26) and (28), respectively. Then, the following estimate holds:*

$$\|u - u_h\|_{0,\Omega} \leq C h^{k+1} |u|_{k+1,\Omega}. \quad (68)$$

Proof. Proceeding exactly as in the proof of Theorem 2, let $w \in H^2(\Omega) \cap H_0^1(\Omega)$ be the solution of the auxiliary problem $-\Delta w = u - u_h$ in Ω, and let $\widetilde{w}_h \in \widetilde{V}_h$ be the finite element approximation of w. Then, from equation (47), integrating by parts and keeping the interelement terms, since u_h is discontinuous, we obtain

$$\|u - u_h\|_{0,\Omega}^2 = \sum_{E\in\mathscr{T}_h} - \int_E \Delta w (u - u_h) = \sum_{E\in\mathscr{T}_h} \left[\int_E \nabla w \cdot \nabla (u - u_h) - \int_{\partial E} \frac{\partial w}{\partial n} (u - u_h) \right]$$
$$= \sum_{E\in\mathscr{T}_h} \left[\int_E \nabla (w - \widetilde{w}_h) \cdot \nabla (u - u_h) + \int_E \nabla \widetilde{w}_h \cdot \nabla (u - u_h) + \int_{\partial E} \frac{\partial w}{\partial n} (u_h - u) \right]. \quad (69)$$

Since $\boldsymbol{r}_e(\widetilde{w}_h) = 0$ in Ω, and u and u_h are the solutions of (26) and (56), respectively, we have

$$\sum_{E\in\mathscr{T}_h} \int_E \nabla (u - u_h) \cdot \nabla \widetilde{w}_h = 0. \quad (70)$$

Hence, (69) reduces to

$$\|u - u_h\|_{0,\Omega}^2 = \sum_{E\in\mathscr{T}_h} \left[\int_E \nabla (w - \widetilde{w}_h) \cdot \nabla (u - u_h) + \int_{\partial E} \frac{\partial w}{\partial n} (u_h - u) \right]. \quad (71)$$

Using Lemma 1, the continuity of u and u_I, and Lemma 4, we obtain

$$\sum_{E\in\mathscr{T}_h} \int_{\partial E} \frac{\partial w}{\partial n} (u_h - u) = \sum_{e\in\mathscr{E}_h} \int_e [\![u_h]\!] \cdot (\nabla w)^0 = \sum_{e\in\mathscr{E}_h} \int_e [\![u_h - u_I]\!] \cdot (\nabla w)^0$$
$$\leq C \, \|u_h - u_I\|_h \left(\sum_{E\in\mathscr{T}_h} \sum_{e\subset\partial E} s(h_e)^{-1} \|w\|_{2,E}^2 \right)^{1/2} \leq C \, \|u_h - u_I\|_h h^k \|w\|_{2,\Omega}, \quad (72)$$

where the last inequality follows from (58). Then, substituting (72) in (71), and using (50) we obtain

$$\|u - u_h\|_{0,\Omega}^2 \leq C \left(\| u - u_h \|_h \, h \, |w|_{2,\Omega} + \| u_h - u_I \|_h \, h^k \, \|w\|_{2,\Omega} \right). \tag{73}$$

The result follows then from (73), (48), and (62). ■

References

[1] S. Agmon, *Lectures on elliptic boundary value problems*, Van Nostrand Mathematical Studies, Princeton, NJ, 1965.

[2] D. N. Arnold, *An interior penalty finite element method with discontinuous element*, SIAM J. Numer. Anal., 19 (1982), pp. 742–760.

[3] G. A. Baker, *Finite element methods for elliptic equations using nonconforming elements*, Math. Comput., 31 (1977), pp. 45–59.

[4] F. Bassi, G. Mariotti, S. Pedinotti, and S. Rebay, *Recent developments on the solution of the full Navier-Stokes equations using a discontinuous finite element method*, in Proceedings of the Ninth International Conference on "Finite Elements in Fluids, New Trend and Applications", M. Morandi Cecchi et al., ed., vol. 1, Venice, October 15–21 1995, pp. 529–539.

[5] F. Bassi and S. Rebay, *Discontinuous finite element high order accurate numerical solution of the compressible Navier-Stokes equations*, in Proceedings of the Conference "Numerical methods for fluid dynamics V", K. W. Morton et al., ed., Oxford: Clarendon Press, April 3–6 1995, pp. 295–302.

[6] ——, *High-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations*, J. Comput. Phys., 131 (1997), pp. 267–279.

[7] F. Bassi, S. Rebay, M. Savini, G. Mariotti, and S. Pedinotti, *A high-order accurate discontinuous finite element method for inviscid and viscous turbomachinery flow*, in Proceedings of the Second European Conference on Turbomachinery Fluid Dynamics and Thermodynamics, Antwerpeen, Belgium, 1997.

[8] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods*, Springer Verlag, Berlin, 1992.

[9] F. Brezzi, G. Manzini, D. Marini, P. Pietra, and A.Russo, *Discontinuous finite elements for diffusion problems.* to appear in Atti del Convegno in Memoria di F. Brioschi, Milano, October 22–23, 1997, Istituto Lombardo di Scienze e Lettere.

[10] P. G. Ciarlet, *The finite element method for elliptic problems*, North-Holland Publishing Company, Amsterdam, Holland, 1980.

[11] B. Cockburn, S. Hou, and C. W. Shu, *TVB Runge-Kutta local projection P^1–discontinuous Galerkin finite element method for conservation laws IV: The multi-dimensional case*, Math. Comp., 54 (1990), pp. 545–581.

[12] B. Cockburn and C. W. Shu, *TVB Runge-Kutta local projection P^1–discontinuous Galerkin finite element method for conservation laws II: General framework*, Math. Comp., 52 (1989), pp. 411–453.

[13] ——, *The local discontinuous Galerkin method for time-dependent convection diffusion problems*, SIAM J. Numer. Anal., 35 (1998), pp. 2440–2463.

[14] J. Douglas, Jr, *H^1–Galerkin methods for a nonlinear Dirichlet problem*, in Proceedings of the Conference "Mathematical Aspects of the Finite Element Methods", Springer Lecture Notes in Mathematics vol. 606, Rome, 1977, pp. 64–86.

[15] J. Douglas, Jr and T. Dupont, *Interior penalty procedures for elliptic and parabolic Galerkin Method*, Lectures Notes in Physics 58, Springer–Verlag, Berlin, 1976.

[16] T. Dupont and R. Scott, *Polynomial approximation of functions in Sobolev spaces*, Math. Comp., 34 (1980), pp. 441–463.

[17] C. Hirsch, *Numerical Computation of Internal and External Flows*, J. Wiley & Sons Ltd., Baffins Lane, Chichester, West Sussex PO19 1UD, England, 1990.

[18] J. L. Lions and E. Magenes, *Problèmes aux Limites non Homogènes et Applications*, vol. 1, Dunod, Paris, 1968.

[19] M. F. Wheeler, *An elliptic collocation-finite element method with interior penalties*, SIAM J. Numer. Anal., 15 (1978), pp. 152–161.