# A survey on DG methods for elliptic problems

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**Abstract.** We present a formulation for elliptic problems that includes all the Discontinuous Galerkin approximations actually present in the literature.

## 1 Introduction

The results here presented are mostly contained in a paper with D.N. Arnold, F. Brezzi, and B. Cockburn [1], which we refer to for a detailed derivation and analysis of the formulation. Let us consider, for simplicity, the model problem

$$-\Delta u = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \Gamma = \partial\Omega. \tag{1}$$

The starting point to derive DG approximations for (1) is to introduce the flux $\boldsymbol{\sigma} = \nabla u$ as an independent variable, so that problem (1) becomes

$$\boldsymbol{\sigma} - \nabla u = 0 \quad \text{in } \Omega; \qquad -\operatorname{div} \boldsymbol{\sigma} = f \quad \text{in } \Omega; \qquad u = 0 \quad \text{on } \Gamma, \tag{2}$$

and then to write a suitable variational formulation for (2) using discontinuous elements for both unknowns $\boldsymbol{\sigma}$ and $u$. To fix ideas, let $\mathscr{T}_h$ be a decomposition of $\Omega$ into triangles $E$. We set, for $k \geq 1$,

$$V_h = \left\{ v \in L^2(\Omega) \text{ such that } v_{|E} \in \mathbb{P}_k(E)\ \forall E \in \mathscr{T}_h \right\},$$
$$\boldsymbol{\Sigma}_h = \left\{ \boldsymbol{\tau} \in \left[L^2(\Omega)\right]^2 \text{ such that } \boldsymbol{\tau}_{|E} \in [\mathbb{P}_k(E)]^2\ \forall E \in \mathscr{T}_h \right\}.$$

Multiplying the first equation of (2) by $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$, the second equation by $v \in V_h$, and integrating by parts we obtain

$$\begin{cases} \displaystyle\int_E \boldsymbol{\sigma} \cdot \boldsymbol{\tau} = -\int_E u \operatorname{div} \boldsymbol{\tau} + \int_{\partial E} u\, \boldsymbol{\tau} \cdot \mathbf{n} & \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\ \forall E \in \mathscr{T}_h, \\ \displaystyle\int_E \boldsymbol{\sigma} \cdot \nabla v = \int_E f v + \int_{\partial E} v\, \boldsymbol{\sigma} \cdot \mathbf{n} & \forall v \in V_h,\ \forall E \in \mathscr{T}_h. \end{cases} \tag{3}$$

In the following, starting from these two equations, we shall write the flux formulation and present two families of numerical fluxes, that lead to two families of formulations. Next, we shall set a unified variational formulation including all the DG methods present in the literature till now. Finally, we shall conclude with error estimates for various DG methods.

## 2 Flux formulation

Summing (3) over $E$ we obtain a first flux formulation:

$$\begin{cases} \text{find } u_h \in V_h,\ \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h \text{ such that:} \\ \displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = -\sum_E \int_E u_h \operatorname{div} \boldsymbol{\tau} + \sum_E \int_{\partial E} \hat{u}\, \boldsymbol{\tau} \cdot \mathbf{n} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\ \displaystyle\sum_E \int_E \boldsymbol{\sigma}_h \cdot \nabla v = \int_\Omega f v + \sum_E \int_{\partial E} v\, \hat{\boldsymbol{\sigma}} \cdot \mathbf{n} \quad \forall v \in V_h, \end{cases} \tag{4}$$

where $\hat{u} = \hat{u}(u_h)$, $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h)$ are the numerical fluxes on the edges, to be defined in order to produce approximations to $u_{|\partial E}$, and $\nabla u_{|\partial E}$, respectively. Denoting by $\mathcal{E}_h$ the set of all edges, and by $\mathcal{E}_h'$ the set of internal edges of $\mathscr{T}_h$, the numerical fluxes are said to be *consistent* if

$$\hat{u}(v) = v_{|\mathcal{E}_h} \quad \text{and} \quad \hat{\boldsymbol{\sigma}}(v, \nabla v) = \nabla v_{|\mathcal{E}_h},\ \forall v \text{ regular}.$$

Moreover, they are said to be *conservative* if they are single-valued on each $e \in \mathcal{E}_h$. This conservativity property is mostly desired for the flux $\hat{\boldsymbol{\sigma}}$, while is less important for $\hat{u}$. Our aim is to eliminate $\boldsymbol{\sigma}_h$ in (4) and to write a variational formulation in the unknwon $u_h$ only. For this, let $E^+$, $E^-$ be two elements sharing an edge $e$, and let $\mathbf{n}^+$, $\mathbf{n}^-$ be the outward unit normal vector to $E^+$, $E^-$, respectively. We denote by $\{\cdot\}$ and $[\![\cdot]\!]$ average and jump on $e$, defined as

$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h';$$
$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h'.$$

On the boundary edges we only need $[\![v]\!]$, $\{\boldsymbol{\tau}\}$, and we simply take $[\![v]\!] = v\mathbf{n}$; $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$.

Let now $q$ be an edge-wise smooth scalar, and let $\boldsymbol{\tau}$ be an edge-wise smooth vector. We obviously accept that they have different values on the two sides of the same edge. Then, using the above definitions of jump $[\![\cdot]\!]$ and average $\{\cdot\}$, we have

$$\sum_E \int_{\partial E} q\, \boldsymbol{\tau} \cdot \mathbf{n} = \sum_e \int_e [\![q]\!] \cdot \{\boldsymbol{\tau}\} + \sum_{e'} \int_{e'} \{q\} [\![\boldsymbol{\tau}]\!], \tag{6}$$

where $e$ ranges over all edges and $e'$ ranges over internal edges. Using the classical Green formula

$$-\int_\Omega u_h \operatorname{div}_h \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} - \sum_E \int_{\partial E} u_h \boldsymbol{\tau} \cdot \mathbf{n},$$

in the first equation of (4), and then (6), we can rewrite problem (4) as

$$
\begin{cases}
\text{find } u_h \in V_h,\ \boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h \text{ such that } \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h,\ \forall v \in V_h \\
\displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} = \int_\Omega \nabla_h u_h \cdot \boldsymbol{\tau} + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \boldsymbol{\tau} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\boldsymbol{\tau}]\!], \\
\displaystyle\int_\Omega \boldsymbol{\sigma}_h \cdot \nabla_h v = \int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\}[\![\hat{\boldsymbol{\sigma}}]\!].
\end{cases}
\tag{8}
$$

We make now the assumption (verified with our choice of $V_h$ and $\boldsymbol{\Sigma}_h$) that

$$\nabla(V_h) \subset \boldsymbol{\Sigma}_h.$$

Then we can take $\boldsymbol{\tau} = \nabla_h v$ in the first equation, and substitute in the second equation, thus obtaining

$$
\begin{aligned}
&\int_\Omega \nabla_h u_h \cdot \nabla_h v + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\nabla_h v]\!] = \\
&\int_\Omega f v + \sum_e \int_e [\![v]\!] \cdot \{\hat{\boldsymbol{\sigma}}\} + \sum_{e'} \int_{e'} \{\hat{v}\}[\![\hat{\boldsymbol{\sigma}}]\!].
\end{aligned}
\tag{10}
$$

It remains to express $\hat{u}$ and $\hat{\boldsymbol{\sigma}}$ in terms of $u_h$. This is easy if we take $\hat{u} = \hat{u}(u_h)$ and $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \nabla_h u_h)$ (this is the choice that characterizes the so-called *primal methods*). However, most *flux methods* take $\hat{u} = \hat{u}(u_h)$ but $\hat{\boldsymbol{\sigma}} = \hat{\boldsymbol{\sigma}}(u_h, \boldsymbol{\sigma}_h)$. In this case we need some additional work, as we shall see later on.

### 2.1 Fluxes depending on $\nabla_h u_h$

A first choice of the fluxes is:

- $\hat{u} = \{u_h\}$ on $e'$, $\hat{u} = 0$ on $e \subset \partial\Omega$,
- $\hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\}$ on every edge. (11)

Notice that this choice implies

$$[\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0, \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\}.$$

Substituting in (10) and rearranging terms, we have

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} = \int_\Omega f v, \tag{12}$$

where we recognize the nonstabilized version of IP method (see, e.g., [10], [2]). We shall list now possible variants of the IP formulation, based on different choices of the fluxes. The first one corresponds to introducing a *stabilizing term*, by taking

$$\begin{aligned} &\bullet\quad \hat{u} = \{u_h\} \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega \\ &\bullet\quad \hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!] \text{ on every edge} \end{aligned} \tag{13}$$

so that

$$[\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0, \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!].$$

Substituting (13) in (10) gives

$$\begin{aligned} &\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} \\ &+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v, \end{aligned} \tag{14}$$

and we obtain the stabilized IP method ([10], [2]).
Generalizing the definition of average by setting, for any real $\beta$,

$$\{v\}_\beta = \beta v^+ + (1-\beta) v^-, \tag{15}$$

with a similar definition for the average of vectors we can define the fluxes as

$$\begin{aligned} &\bullet\quad \hat{u} = \{u_h\}_{(1-\beta)} \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega \\ &\bullet\quad \hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\}_\beta - c\,|e|^{-1}[\![u_h]\!] \text{ on every edge} \end{aligned} \tag{16}$$

so that

$$\begin{aligned} [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = \{u_h\}_{(1-\beta)} - \{u_h\}, \\ [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\}_\beta - c\,|e|^{-1}[\![u_h]\!]. \end{aligned}$$

Substituting (16) in (10) gives

$$\begin{aligned} &\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\}_\beta - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\}_\beta \\ &+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v, \end{aligned} \tag{17}$$

and we obtain the method of Heinrich [11].
Choosing, on the boundary of each element $E$,

$$\begin{aligned} &\bullet\quad \hat{u} = \{u_h\} + \mathbf{n}_E [\![u_h]\!] \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega \\ &\bullet\quad \hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} \text{ on every edge} \end{aligned} \tag{18}$$

for which we have

$$[\![\hat{u} - u_h]\!] = -[\![u_h]\!] + 2[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0, \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\},$$

leads, after substitution in (10), to the method of Baumann and Oden [6]:

$$\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} + \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} = \int_\Omega f v. \quad (19)$$

Finally, taking, on the boundary of each element $E$,

$$\begin{aligned} &\bullet\ \hat{u} = \{u_h\} + \mathbf{n}_E [\![u_h]\!] \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega \\ &\bullet\ \hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\,|e|^{-1} [\![u_h]\!] \text{ on every edge} \end{aligned} \quad (20)$$

for which we have

$$[\![\hat{u} - u_h]\!] = -[\![u_h]\!] + 2[\![u_h]\!], \ \ \{\hat{u} - u_h\} = 0 [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \ \ \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,|e|^{-1}[\![u_h]\!],$$

gives, always after substitution in (10),

$$\begin{aligned} &\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} + \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} \\ &+ \sum_e c\,|e|^{-1} \int_e [\![u_h]\!] \cdot [\![v]\!] = \int_\Omega f v, \end{aligned} \quad (21)$$

that is, the so-called NIPG method [12], which is the stabilized version of (19). Let us now turn to cases where $\hat{\boldsymbol{\sigma}}$ depends on $\boldsymbol{\sigma}_h$.

### 2.2 Fluxes depending on $\boldsymbol{\sigma}_h$

When the flux $\hat{\boldsymbol{\sigma}}$ depends on $\boldsymbol{\sigma}_h$, the elimination of $\boldsymbol{\sigma}_h$ from the first equation of (8) and its substitution into the second equation requires some additional work. For this, to any scalar function $v$ we associate the vectors $\boldsymbol{R}([\![v]\!])$, $l(\{v\})$ defined by:

$$\begin{aligned} &\bullet\ \boldsymbol{R}([\![v]\!]) \in \boldsymbol{\Sigma}_h : \quad \int_\Omega \boldsymbol{R}([\![v]\!]) \cdot \boldsymbol{\tau} = -\sum_e \int_e [\![v]\!] \cdot \{\boldsymbol{\tau}\} \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\ &\bullet\ l(\{v\}) \in \boldsymbol{\Sigma}_h : \quad \int_\Omega l(\{v\}) \cdot \boldsymbol{\tau} = -\sum_{e'} \int_{e'} \{v\} [\![\boldsymbol{\tau}]\!] \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h. \end{aligned} \quad (22)$$

Using (22) in the first equation of (10) gives

$$\boldsymbol{\sigma}_h = \nabla_h u_h - \boldsymbol{R}([\![\hat{u} - u_h]\!]) - l(\{\hat{u} - u_h\}),$$

and the second equation of (10) becomes

$$\text{find } u_h \in V_h : \quad B_h(u_h, v) = (f, v) \quad \forall v \in V_h$$

where:

$$
\begin{aligned}
B_h(u_h, v) :=& \int_\Omega \nabla_h u_h \cdot \nabla_h v - \int_\Omega \boldsymbol{R}([\![\hat{u} - u_h]\!]) \cdot \nabla_h v \\
& - \int_\Omega l(\{\hat{u} - u_h\}) \cdot \nabla_h v - \sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] - \sum_{e'} \int_{e'} [\![\hat{\boldsymbol{\sigma}}]\!]\{v\} \\
& \equiv \int_\Omega \nabla_h u_h \cdot \nabla_h v + \sum_e \int_e [\![\hat{u} - u_h]\!] \cdot \{\nabla_h v\} \\
& + \sum_{e'} \int_{e'} \{\hat{u} - u_h\}[\![\nabla_h v]\!] - \sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] - \sum_{e'} \int_{e'} [\![\hat{\boldsymbol{\sigma}}]\!]\{v\}.
\end{aligned}
\tag{23}
$$

We will indicate now various choices for the fluxes in (23). Choosing

$$
\begin{aligned}
&\bullet\quad \hat{u} = \{u_h\} \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega, \\
&\bullet\quad \hat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_h\} \text{ on every edge,}
\end{aligned}
\tag{24}
$$

we observe that

$$
[\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0, \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\boldsymbol{\sigma}_h\}.
$$

Consequently, $\boldsymbol{\sigma}_h = \nabla_h u_h + \boldsymbol{R}([\![u_h]\!])$, and $\boldsymbol{R}([\![u_h]\!])$ is a sort of strain correction due to the discontinuity of the approximation. Therefore,

$$
\begin{aligned}
\sum_e \int_e \{\hat{\boldsymbol{\sigma}}\} \cdot [\![v]\!] =& \sum_e \int_e \{\nabla_h u_h\} \cdot [\![v]\!] + \sum_e \int_e \{\boldsymbol{R}([\![u_h]\!])\} \cdot [\![v]\!] \\
=& - \int_\Omega \nabla_h u_h \cdot \boldsymbol{R}([\![v]\!]) - \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!]).
\end{aligned}
$$

Substituting in (23) gives the first Bassi-Rebay formulation [4]:

$$
u_h \in V_h : \quad \int_\Omega [\nabla_h u_h + \boldsymbol{R}([\![u_h]\!])] \cdot [\nabla_h v + \boldsymbol{R}([\![v]\!])] = \int_\Omega f v \quad \forall v \in V_h, \tag{25}
$$

which can be equivalently written as

$$
\begin{aligned}
& \int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e \{\nabla_h u_h\} \cdot [\![v]\!] \\
& - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} + \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!]) = \int_\Omega f v \quad \forall v \in V_h.
\end{aligned}
\tag{26}
$$

Unfortunately, formulation (25) is not stable, as shown in [7], but it can be stabilized by introducing the strain correction on each edge:

$$
\bullet\ v \longrightarrow \boldsymbol{r}_e([\![v]\!]) \in \boldsymbol{\Sigma}_h : \quad \int_\Omega \boldsymbol{r}_e([\![v]\!]) \cdot \boldsymbol{\tau} + \int_e [\![v]\!] \cdot \{\boldsymbol{\tau}\} = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \tag{27}
$$

(and notice that $\sum_e \boldsymbol{r}_e([\![v]\!]) = \boldsymbol{R}([\![v]\!])$). The second Bassi-Rebay formulation (introduced in [5] and proved to be stable in [7]) is:

$$\begin{aligned} &\int_\Omega [\nabla_h u_h + \boldsymbol{R}([\![u_h]\!])] \cdot [\nabla_h v + \boldsymbol{R}([\![v]\!])] \\ &- \int_\Omega \boldsymbol{R}([\![u_h]\!]) \cdot \boldsymbol{R}([\![v]\!]) + c \sum_e \int_\Omega \boldsymbol{r}_e([\![u_h]\!]) \cdot \boldsymbol{r}_e([\![v]\!]) = \int_\Omega fv \quad \forall v \in V_h. \end{aligned} \tag{28}$$

It can be easily seen that (28) corresponds to taking the fluxes in (23) as

$$\begin{aligned} &\bullet \quad \hat{u} = \{u_h\} \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega, \\ &\bullet \quad \hat{\boldsymbol{\sigma}} = \{\nabla_h u_h\} - c\,\boldsymbol{r}_e([\![u_h]\!]) \text{ on every edge,} \end{aligned} \tag{29}$$

so that

$$[\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = 0, \quad [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\nabla_h u_h\} - c\,\{\boldsymbol{r}_e([\![u_h]\!])\}.$$

Then we have from (23)

$$\begin{aligned} &\int_\Omega \nabla_h u_h \cdot \nabla_h v - \sum_e \int_e [\![v]\!] \cdot \{\nabla_h u_h\} - \sum_e \int_e [\![u_h]\!] \cdot \{\nabla_h v\} \\ &+ c \sum_e \int_\Omega \boldsymbol{r}_e([\![u_h]\!]) \cdot \boldsymbol{r}_e([\![v]\!]) = \int_\Omega fv, \end{aligned}$$

and we see that the difference with IP is only in the choice of the stabilizing term. As we have seen for the IP formulation, many variants are possible also for the first Bassi-Rebay formulation. For instance, taking

$$\begin{aligned} &\bullet \quad \hat{u} = \{u_h\}_{(1-\beta)} \text{ on } e', \quad \hat{u} = 0 \text{ on } e \subset \partial\Omega, \\ &\bullet \quad \hat{\boldsymbol{\sigma}} = \{\boldsymbol{\sigma}_h\}_\beta - c\,|e|^{-1}[\![u_h]\!] \text{ on every edge,} \end{aligned} \tag{31}$$

for which we have

$$\begin{aligned} [\![\hat{u} - u_h]\!] = -[\![u_h]\!], \quad \{\hat{u} - u_h\} = \{u_h\}_{(1-\beta)} - \{u_h\}, \\ [\![\hat{\boldsymbol{\sigma}}]\!] = 0, \quad \{\hat{\boldsymbol{\sigma}}\} = \{\boldsymbol{\sigma}_h\}_\beta - c\,|e|^{-1}[\![u_h]\!], \end{aligned}$$

where $\{v\}_\beta$ is taken as in (15), we obtain the most widely used variant of the LDG method of Cockburn-Shu [9]. For other variants we refer to [1].

## 3 Congergence properties

We give in this section a hint of error estimates that can be obtained. As usual, we need boundedness, stability and consistency in a suitable norm. Define the space:

$$V(h) = V_h + H^2(\Omega) \cap H^1_0(\Omega) \subset H^2(\mathscr{T}_h),$$

with the norm:

$$|||v|||^2 = |v|^2_{1,h} + \Sigma_E h_E^2 |v|^2_{2,h} + \sum_e ||\boldsymbol{r}_e([\![v]\!])||^2_{0,\Omega} \quad v \in V(h),$$

and note that $\sum_e ||\boldsymbol{r}_e([\![v]\!])||^2_{0,\Omega} \simeq \sum_e h_e^{-1}||[\![v]\!]||^2_{0,e} \quad v \in V(h)$. We also need classical approximation properties:

$$|||u - u_I||| \le Ch^k |u|_{k+1,\Omega},$$

where $u_I$ denotes a suitable interpolant of $u$ verifying:

$$|u - u_I|_{s,E} \le Ch_E^{k+1-s}|u|_{k+1,E} \quad s = 0, 1, 2.$$

Then,

$$\begin{aligned} C_s|||u_I - u_h|||^2 &\le B_h(u_I - u_h, u_I - u_h) = B_h(u_I - u, u_I - u_h) \\ &\le C_b|||u_I - u_h|||\,|||u_I - u||| \le Ch^k|u|_{k+1,\Omega}|||u_I - u_h|||. \end{aligned}$$

To obtain optimal $L^2$ estimates we need the "adjoint consistency" property: given a problem $\mathcal{A}u = f$, that we approximate with

$$B_h(u_h, v_h) = (f, v_h) \quad \forall v_h \in V_h,$$

the consistency property amounts to require that

$$B_h(u, v_h) = (f, v_h) \quad \forall v_h \in V_h.$$

The adjoint consistency property requires that for every adjoint problem $\mathcal{A}^*\psi = g$ one has

$$B_h^*(\psi, v_h) \equiv B_h(v_h, \psi) = (g, v_h) \quad \forall v_h \in V_h.$$

Taking $g = u - u_h$ he have then

$$\begin{aligned} ||u - u_h||_0^2 &= (u - u_h, u - u_I) + (u - u_h, u_I - u_h) \\ &= (u - u_h, u - u_I) + B_h(u_I - u_h, \psi) \\ &= (u - u_h, u - u_I) + B_h(u_I - u_h, \psi - \psi_h) \\ &\le ||u - u_h||_0||u - u_I||_0 + Ch|\psi|_{2,\Omega}|||u_I - u_h|||. \end{aligned}$$

In Table 1 we report the choice of fluxes for various DG methods. By $\alpha_j$ and $\alpha_r$ we indicate the type of stabilization: through the jumps (as in (14)) or through the $\boldsymbol{r}_e$ (resp.), as in (28). In Table 2 we collect the properties and the order of convergence of the various methods in $H^1$ and $L^2$ (cons. stands for consistency, a.c. for adjoint consistency, stab. for stability, type is the type of stabilization, and cond. indicates the condition on the coefficient of the stabilization term). Most of the methods listed in the Tables were recalled in this paper. For the others we provide the reference.

**Table 1.** Choice of the numerical fluxes

| Method | $\widehat{u}$ | $\widehat{\sigma}$ |
|---|---|---|
| IP [10] | $\{u_h\}$ | $\{\nabla_h u_h\} - \alpha_j$ |
| BO [6] | $\{u_h\} + \mathbf{n}_E \cdot [u_h]$ | $\{\nabla_h u_h\}$ |
| NIPG [12] | $\{u_h\} + \mathbf{n}_E \cdot [u_h]$ | $\{\nabla_h u_h\} - \alpha_j$ |
| H[11] | $\{u_h\}_{(1-\beta)}$ | $\{\nabla_h u_h\}_\beta - \alpha_j$ |
| BZ[3] | $u_{h\vert E}$ | $-\alpha_j$ |
| BR 1[4] | $\{u_h\}$ | $\{\boldsymbol{\sigma}_h\}$ |
| BR 2 | $\{u_h\}$ | $\{\nabla_h u_h\} - \alpha_r$ |
| BMMPR1[7] | $\{u_h\}$ | $\{\boldsymbol{\sigma}_h\} - \alpha_r$ |
| LDG [9] | $\{u_h\}_{(1-\beta)}$ | $\{\boldsymbol{\sigma}_h\}_\beta - \alpha_j$ |
| BMMPR2[8] | $u_{h\vert E}$ | $-\alpha_r$ |

**Table 2.** Properties of various DG methods

| Method | cons. | a.c. | stab. | type | cond. | $H^1$ | $L^2$ |
|---|---|---|---|---|---|---|---|
| IP [10] | ✓ | ✓ | ✓ | $\alpha_j$ | $c > c^*$ | $h^k$ | $h^{k+1}$ |
| BO[6] $(k \geq 2)$ | ✓ | × | ✓ | - | – | $h^k$ | $h^k$ |
| NIPG [12] | ✓ | × | ✓ | $\alpha_j$ | $c > 0$ | $h^k$ | $h^k$ |
| H [11] | ✓ | ✓ | ✓ | $\alpha_j$ | $c > c^*$ | $h^k$ | $h^{k+1}$ |
| BZ [3] | × | × | ✓ | $\alpha_j$ | $c \approx h^{-2k-1}$ | $h^k$ | $h^{k+1}$ |
| BR 1 [4] | ✓ | ✓ | × | - | - | $[h^k]$ | $[h^{k+1}]$ |
| BR 2 [8] | ✓ | ✓ | ✓ | $\alpha_r$ | $c > 3$ | $h^k$ | $h^{k+1}$ |
| BMMPR1 [7] | ✓ | ✓ | ✓ | $\alpha_r$ | $c > 0$ | $h^k$ | $h^{k+1}$ |
| LDG [9] | ✓ | ✓ | ✓ | $\alpha_j$ | $c > 0$ | $h^k$ | $h^{k+1}$ |
| BMMPR2 [8] | × | × | ✓ | $\alpha_r$ | $c \approx h^{-2k}$ | $h^k$ | $h^{k+1}$ |

We conclude with an explanation on the brackets used in the error estimates of Table 2 for the Bassi-Rebay formulation. We have:

$$||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\Omega} \leq Ch^k |u|_{k+1,\Omega}, \quad ||P_{k-1}(u - u_h)||_{0,\Omega} \leq Ch^{k+1}|u|_{k+1,\Omega}, \quad (35)$$

where $\boldsymbol{\sigma} = -\nabla u$, $\boldsymbol{\sigma}_h = -\nabla_h u_h - \boldsymbol{R}([\![u_h]\!])$, and $P_{k-1}$ is the projection operator (element by element) on the space of polynomials of degree $\leq k-1$.

For more detailed estimates on specific methods or classes of methods we refer to [1] and the references therein.

## 4 Conclusions

The present framework allows an easy comparison of various DG methods for diffusive problems (or for the diffusive part of advection-diffusion problems), as it includes all existing methods. Primal formulations are easier to deal with, and, in general, produce a smaller stencil. On the other hand, flux formulations are more in the spirit of the methods for dealing with convective terms and/or purely hyperbolic problems. Much work, and much more numerical experiments are still to be done in order to assess the relative merits of the various methods. In particular, the potential of DG methods in connection with Domain Decomposition Methods and $h$, $p$ or $hp$ adaptive algorithms has still to be explored.

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