Discontinuous Galerkin elements for Reissner-Mindlin plates

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Abstract We present an overview of some families of locking-free elements for Reissner-Mindlin plates recently introduced and analyzed in [2] and [1]. They are all based on the ideas of discontinuous Galerkin approach, and they vary in the amount of interelement continuity required.

1 Introduction

The Reissner–Mindlin model for moderately thick clamped plates consists in loocking for the rotation vector θ and the transverse displacement w which minimize over $H^1_0(\Omega)\times H^1_0(\Omega)$ the (scaled) plate energy

$$J(\theta,w)=\frac{1}{2}\int_\Omega \mathsf{C}\,\varepsilon(\theta):\varepsilon(\theta)\,\mathrm{d}x+\frac{1}{2}\lambda t^{-2}\int_\Omega |\nabla w-\theta|^2\,\mathrm{d}x-\int_\Omega gw\,\mathrm{d}x, \qquad (1)$$

where the coefficients C and λ depend on the material properties of the plate, g is the scaled load, and t is the plate thickness. If one minimizes the energy over subspaces consisting of low order finite elements, then the resulting approximation suffers from the problem of *locking*, which can be described as follows. As t tends to 0, the solution of (1) tends to (θ_0,w_0), where $\theta_0=\nabla w_0$ which, in general, will not be zero (actually, w_0 will be the solution of the Kirchhoff model). If we discretize the problem directly by seeking $\theta_h\in\Theta_h$ and $w_h\in W_h$ minimizing $J(\theta,w)$ over $\Theta_h\times W_h$, then as t vanishes, (θ_h,w_h) will converge to some $(\theta_{0,h},w_{0,h})$ where, again, $\theta_{0,h}=\nabla w_{0,h}$. For low order finite element spaces, this last condition is too restrictive. In particular, if continuous piecewise linear functions are used to approximate both variables, then $\theta_{0,h}\equiv\nabla w_{0,h}$ would be continuous *and* piecewise constant, with zero boundary conditions. Only the choice $\theta_{0,h}=0$ can satisfy all these conditions. For

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t very small, the quantity $\theta_h - \nabla w_h$, although not necessarily zero, must be very small, and hence θ_h will be very close to zero, instead of being close to θ which, in turn, will be close to θ_0. Another way of looking at this problem is from the point of view of approximation: for small t, one cannot find suitable interpolants θ^I and w^I that are close to θ and w, respectively, if one requires $\theta^I - \nabla w^I$ to be of the order of t^2.

A number of approaches have been developed to avoid the locking problem. One successful idea has been to introduce an additional finite element space Γ_h and a reduction operator $P_h : \Theta_h \to \Gamma_h$, and then look for $\theta_h \in \Theta_h$ and $w_h \in W_h$ minimizing a modified energy functional

$$J_h(\theta, w) = \frac{1}{2}\int_\Omega \mathsf{C}\,\varepsilon(\theta) : \varepsilon(\theta)\,\mathrm{d}x + \frac{1}{2}\lambda t^{-2}\int_\Omega |\nabla w - P_h\theta|^2\,\mathrm{d}x - \int_\Omega g w\,\mathrm{d}x. \quad (2)$$

A crucial assumption is that ∇W_h is a subset of Γ_h, and in particular of the image of P_h. As t tends to 0, the limiting condition will now be the much less demanding

$$P_h\theta_{0,h} = \nabla w_{0,h}. \quad (3)$$

Various locking-free finite elements have been obtained in this way (see, e.g., [3], [5], [8], [11], [12], [9], [13], [10]).

In [2], the techniques of Discontinuous Galerkin (DG) methods were used to develop two families of odd-degree locking-free elements. Since DG solutions are not required to satisfy the standard interelement continuity conditions of conforming finite element methods (that is, continuous elements in the case of the Reissner–Mindlin plate problem), the method allows a greater flexibility.

Starting from the approach of [2], other elements were introduced ([9], [13], [10]) for the functional (2), while in [1] a collection of families of locking-free elements which do not need the reduction operator P_h were developed. The common feature in all the methods considered in [1] is to choose W_h to be piecewise polynomials of degree $\leq k$ (with $k \geq 2$), and $\Theta_h = \Gamma_h$ to be piecewise polynomials of degree $\leq k-1$. The methods vary in the amount of interelement continuity required.

In the present paper we shall give an overview of some DG elements, and we shall report the convergence results, referring for the proofs to the corresponding papers.

2 Discontinuous Galerkin discretization

Introducing the shear stress $\gamma = \lambda t^{-2}(\nabla w - \theta)$ as an auxiliary variable, and writing the Euler equations for the energy functional (1) we may write the Reissner–Mindlin equations as:

$$-\operatorname{div}\mathsf{C}\,\varepsilon(\theta)-\gamma=0 \quad \text{in } \Omega, \tag{4}$$
$$-\operatorname{div}\gamma=g \quad \text{in } \Omega, \tag{5}$$
$$\nabla w-\theta-t^2\gamma=0 \quad \text{in } \Omega, \tag{6}$$
$$\theta=0,\ w=0 \text{ on } \partial\Omega. \tag{7}$$

Equation (6) should actually be $\nabla w-\theta-\lambda^{-1}t^2\gamma=0$, where λ is the *shear correction factor*, but we set $\lambda=1$ to simplify the presentation. By setting

$$a(\theta,\eta)=(\mathsf{C}\,\varepsilon(\theta),\varepsilon(\eta)) \quad \text{for } \theta,\,\eta\in H^1(\Omega)$$

the variational formulation of equations (4)–(7) is:
Given $g\in L^2(\Omega)$, find $\theta\in H_0^1(\Omega)$, $w\in H_0^1(\Omega)$ and $\gamma\in L^2(\Omega)$ such that

$$a(\theta,\eta)+(\gamma,\nabla v-\eta)=(g,v) \quad \forall(\eta,v)\in H_0^1(\Omega)\times H_0^1(\Omega), \tag{8}$$
$$(\nabla w-\theta,\tau)-t^2(\gamma,\tau)=0 \quad \forall\tau\in L^2(\Omega). \tag{9}$$

Before proceeding we need to introduce some notations. We shall use the usual Sobolev spaces such as $H^s(\Omega)$, with the corresponding seminorm and norm denoted by $|\cdot|_s$ and $\|\cdot\|_s$, respectively. By convention, we use boldface type for the vector-valued analogues ($H^s(\Omega)=[H^s(\Omega)]^2$), and calligraphic type for symmetric-tensor-valued analogues ($\mathscr{H}^s(\Omega)=[H^s(\Omega)]^2_{\mathrm{sym}}$); we use parentheses $(\cdot,\cdot)$ to denote the inner product in any of the spaces $L^2(\Omega)$, $L^2(\Omega)$, or $\mathscr{L}^2(\Omega)$.
We recall the following result (see [3], [4] for a more general case). If Ω is a convex polygonal domain, and C is smooth, then problem (8)–(9) has a unique solution that verifies

$$\|\theta\|_2+\|w\|_2+\|\gamma\|_0+t\|\gamma\|_1\le C(\|g\|_{-1}+t\|g\|_0), \tag{10}$$

where C is a constant depending only on Ω and on the coefficients in C.

Let now $\mathscr{T}_h$ be a family of shape-regular decompositions of Ω into triangles T and let $\mathscr{E}_h$ denote the set of all the edges in $\mathscr{T}_h$. For piecewise polynomial spaces, we use the notation

$$\mathscr{L}_k^s(\mathscr{T}_h)=\{\,v\in H^s(\Omega)\,:\,v|_T\in\mathscr{P}_k(T),\,T\in\mathscr{T}_h\,\}, \tag{11}$$

where, as usual, $\mathscr{P}_k(T)$ is the set of polynomials of degree at most k on T. Since we will work with discontinuous finite elements not belonging to $H^1(\Omega)$, we define the space

$$H^1(\mathscr{T}_h):=\{\,v\in L^2(\Omega)\,:\,v|_T\in H^1(T),\,T\in\mathscr{T}_h\,\}. \tag{12}$$

Differential operators can be applied to this space only piecewise. We indicate this by a subscript h on the operator. Hence, the space $H^1(\mathscr{T}_h)$ will be equipped with the seminorm $|v|_{1,h}=\|\nabla_h v\|_0$ and the corresponding norm $\|v\|_{1,h}^2=|v|_{1,h}^2+\|v\|_0^2$.

Finally, before deriving a DG discretization of (8)–(9) we need to introduce typical tools as *averages* and *jumps* on the edges of $\mathscr{T}_h$. Let e be an internal edge of $\mathscr{T}_h$, shared by two elements T^+ and T^-, and let n^+ and n^- denote the unit normals to e, pointing outward from T^+ and T^-, respectively. If φ belongs to $H^1(\mathscr{T}_h)$ (or possibly the vector- or tensor-valued analogue), we define the average $\{\varphi\}$ on e as usual:

$$\{\varphi\} = \frac{\varphi^+ + \varphi^-}{2}.$$

For a scalar function $\varphi \in H^1(\mathscr{T}_h)$ we define its jump on e as

$$[\![\varphi]\!] = \varphi^+ n^+ + \varphi^- n^-,$$

which is a vector normal to e. The jump of a vector $\varphi \in H^1(\mathscr{T}_h)$ is the symmetric matrix-valued function given on e by:

$$[\![\varphi]\!] = \varphi^+ \odot n^+ + \varphi^- \odot n^-,$$

where $\varphi \odot n = (\varphi \otimes n + n \otimes \varphi)/2$ is the symmetric part of the tensor product of φ and n.

On a boundary edge, the average $\{\varphi\}$ is defined simply as the trace of φ, while for a scalar-valued function we define $[\![\varphi]\!]$ to be φn (with n the outward unit normal), and for a vector-valued function we define $[\![\varphi]\!] = \varphi \odot n$.

It is easy to check that, (using the symbol $\langle \cdot\,,\cdot \rangle$ to denote L^2-inner product of functions or vectors on $\mathscr{E}_h$)

$$\sum_{T\in\mathscr{T}_h} \int_{\partial T} \varphi \cdot n_T v \,\mathrm{d}s = \langle \{\varphi\}, [\![v]\!] \rangle, \quad \varphi \in H^1(\Omega), v \in H^1(\mathscr{T}_h). \tag{13}$$

Similarly,

$$\sum_{T\in\mathscr{T}_h} \int_{\partial T} \mathscr{S} n_T \cdot \eta \,\mathrm{d}s = \langle \{\mathscr{S}\}, [\![\eta]\!] \rangle, \quad \mathscr{S} \in \mathscr{H}^1(\Omega), \eta \in H^1(\mathscr{T}_h). \tag{14}$$

To derive a finite element method for the Reissner–Mindlin system based on discontinuous elements, we test (4) against a test function $\eta \in H^2(\mathscr{T}_h)$ and (5) against a test function $v \in H^1(\mathscr{T}_h)$, integrate by parts, and add. Since η and v may be discontinuous across element boundaries, we obtain terms at the interelement boundaries that we manipulate using (13)-(14). We obtain:

$$(\mathsf{C}\varepsilon_h(\theta), \varepsilon_h(\eta)) - \langle \{\mathsf{C}\varepsilon_h(\theta)\}, [\![\eta]\!] \rangle + (\gamma, \nabla_h v - \eta) - \langle \{\gamma\}, [\![v]\!] \rangle = (g, v), \tag{15}$$
$$(\eta, v) \in H^2(\mathscr{T}_h) \times H^1(\mathscr{T}_h),$$
$$(\nabla_h w - \theta, \tau) - t^2(\gamma, \tau) = 0, \qquad \tau \in H^1(\mathscr{T}_h). \tag{16}$$

The second and fourth terms in (15) involve integrals over the edges and would not be present in conforming methods. They arise from the integration by parts and are necessary to maintain consistency.

We now proceed as is common for DG methods. (For a different point of view on this type of derivation see [6]). First, we add terms to symmetrize this formulation so that it is adjoint-consistent as well. Second, to stabilize the method, we add *interior penalty* terms $p_\Theta(\theta, \eta)$ and $p_W(w, v)$ in which the functions p_Θ and p_W will depend only on the jumps of their arguments. Following [2] we set

$$p_\Theta(\theta,\eta) = \sum_{e\in\mathscr{E}_h} \frac{\kappa^\Theta}{|e|}\int_e [\![\theta]\!] : [\![\eta]\!]\,\mathrm{d}s, \quad p_W(w,v) = \sum_{e\in\mathscr{E}_h} \frac{\kappa^W}{|e|}\int_e [\![w]\!]\cdot[\![v]\!]\,\mathrm{d}s, \tag{17}$$

so that $p_\Theta(\eta,\eta)$, $(p_W(v,v)$, respectively) can be viewed as a measure of the deviation of η (v, respectively) from being continuous. The parameters κ^Θ and κ^W are positive constants to be chosen; they must be sufficiently large to ensure stability. Since $[\![\theta]\!]=0$ and $[\![w]\!]=0$, equations (15)-(16) can then be written as

$$\begin{aligned}(\mathsf{C}\,\varepsilon_h(\theta),\varepsilon_h(\eta)) - \langle\{\mathsf{C}\,\varepsilon_h(\theta)\},[\![\eta]\!]\rangle - \langle[\![\theta]\!],\{C\,\varepsilon_h(\eta)\}\rangle\rangle + (\gamma,\nabla_h v-\eta)\\ -\langle\{\gamma\},[\![v]\!]\rangle + p_\Theta(\theta,\eta)+p_W(w,v) = (g,v), \quad (\eta,v)\in H^2(\mathscr{T}_h)\times H^1(\mathscr{T}_h),\end{aligned} \tag{18}$$

$$(\nabla_h w-\theta,\tau) - \langle[\![w]\!],\{\tau\}\rangle - t^2(\gamma,\tau) = 0, \quad \tau\in H^1(\mathscr{T}_h). \tag{19}$$

To obtain a DG discretization, we have to choose finite dimensional subspaces $\Theta_h\subset H^2(\mathscr{T}_h)$, $W_h\subset H^1(\mathscr{T}_h)$, and $\Gamma_h\subset H^1(\mathscr{T}_h)$, and then write the discrete problem:

Find $(\theta_h,w_h)\in\Theta_h\times W_h$ *and* $\gamma_h\in\Gamma_h$ *such that*

$$\begin{aligned}(\mathsf{C}\,\varepsilon_h(\theta_h),\varepsilon_h(\eta)) - \langle\{\mathsf{C}\,\varepsilon_h(\theta_h)\},[\![\eta]\!]\rangle - \langle[\![\theta_h]\!],\{\mathsf{C}\,\varepsilon_h(\eta)\}\rangle\rangle\\ +(\gamma_h,\nabla_h v-\eta) - \langle\{\gamma_h\},[\![v]\!]\rangle\\ +p_\Theta(\theta_h,\eta)+p_W(w_h,v) = (g,v), \quad (\eta,v)\in\Theta_h\times W_h,\end{aligned} \tag{20}$$

$$(\nabla_h w_h-\theta_h,\tau) - \langle[\![w_h]\!],\{\tau\}\rangle - t^2(\gamma_h,\tau) = 0, \quad \tau\in\Gamma_h. \tag{21}$$

For any choice of the finite element spaces Θ_h, W_h, and Γ_h, and any interior penalty functions p_Θ and p_W depending only on the jumps of their arguments, this gives a consistent finite element method since no reduction operator P_h is used. If instead P_h is needed, there will be a consistency error to be estimated, and equations (20)-(21) will be modified into:

$$\begin{aligned}(\mathsf{C}\,\varepsilon_h(\theta_h),\varepsilon_h(\eta)) - \langle\{\mathsf{C}\,\varepsilon_h(\theta_h)\},[\![\eta]\!]\rangle - \langle[\![\theta_h]\!],\{\mathsf{C}\,\varepsilon_h(\eta)\}\rangle\rangle\\ +(\gamma_h,\nabla_h v-P_h\eta) - \langle\{\gamma_h\},[\![v]\!]\rangle\\ +p_\Theta(\theta_h,\eta)+p_W(w_h,v) = (g,v), \quad (\eta,v)\in\Theta_h\times W_h,\end{aligned} \tag{22}$$

$$(\nabla_h w_h-P_h\theta_h,\tau) - \langle[\![w_h]\!],\{\tau\}\rangle - t^2(\gamma_h,\tau) = 0, \quad \tau\in\Gamma_h. \tag{23}$$

In the next section we shall recall different choices of the finite element spaces.

3 The finite elements

We recall in this section some DG-elements/families developed so far. We refer to the original papers for detailed proofs, and we will just recall the resulting error estimates obtained in the DG-norms defined as

$$
\begin{aligned}
|||\eta|||_{\Theta}^2 &:= \|\eta\|_{1,h}^2 + \sum_{e\in\mathscr{E}_h}\left(\frac{1}{|e|}\|[\![\eta]\!]\|_{0,e}^2 + |e|\,\|\{\mathrm{C}\,\varepsilon_h(\eta)\}\|_{0,e}^2\right), \qquad \eta\in H^2(\mathscr{T}_h),\\
|||v|||_{W}^2 &:= |v|_{1,h}^2 + \sum_{e\in\mathscr{E}_h}\frac{1}{|e|}\|[\![v]\!]\|_{0,e}^2, \qquad v\in H^1(\mathscr{T}_h), \qquad (24)\\
|||\tau|||_{\Gamma}^2 &:= \|\tau\|_0^2 + \sum_{e\in\mathscr{E}_h}|e|\|\{\tau\}\|_{0,e}^2, \qquad \tau\in H^1(\mathscr{T}_h).
\end{aligned}
$$

3.1 DG-Elements based on the use of the reduction operator P_h

Example 3.1.1 The following family of elements of odd degree $k \geq 1$ was introduced in [2]:

$$
\Theta_h = \mathscr{L}_k^0(\mathscr{T}_h), \quad W_h = \mathscr{L}_k^0(\mathscr{T}_h), \quad \Gamma_h = \mathscr{L}_{k-1}^0(\mathscr{T}_h), \tag{25}
$$

where $\mathscr{L}_k^0(\mathscr{T}_h)$ denotes the space of discontinuous piecewise polynomials of degree $\leq k$ (see (11)). The penalty term $p_\Theta(\theta,\eta)$ is taken as in (17), while $p_W(w,v)$ is somewhat weaker:

$$
p_W(w,v) = \sum_{e\in\mathscr{E}_h}\frac{\kappa^W}{|e|}\int_e Q_e[\![w]\!]\cdot Q_e[\![v]\!]\,ds, \tag{26}
$$

and Q_e is the projection onto polynomials of degree $k-1$. The error estimates in the norms (24) are:

$$
\begin{aligned}
&|||\theta-\theta_h|||_\Theta + |||w-w_h|||_W + t|||\gamma-\gamma_h|||_\Gamma \qquad (27)\\
&\qquad \leq C h^k\left(\|\theta\|_{k+1,\Omega} + \|w\|_{k+1,\Omega} + t\|\gamma\|_{k,\Omega} + \|\gamma\|_{k-1,\Omega}\right),
\end{aligned}
$$

which are optimal in terms of order of convergence, and for the case $k=1$ also in terms of regularity (see (10)). The definition of P_h is quite complicated and will not be detailed here. We note however that, for the lowest order case $k=1$, the reduction operator P_h is simply the L^2 projection onto the piecewise constant space $\mathscr{L}_0^0(\mathscr{T}_h)$. The degrees of freedom are shown in Fig. 1.

Example 3.1.2 In the spirit of [2], a linear nonconforming element plus a quadratic nonconforming bubble was first obtained and analyzed in [9]. Then Lovadina in [13] showed that the bubble is actually not needed, and also proved optimal L^2-estimates (see also [10]). Denoting by P_1^{nc} the space of piecewise linear polynomials continuous at the midpoint of each edge of $\mathscr{T}_h$, the choice of spaces is

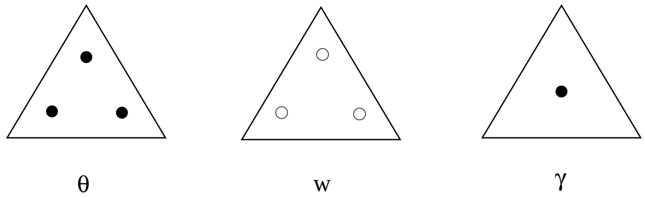

Fig. 1 Totally discontinuous elements: d.o.f. for the lowest order case

$$\Theta_h = P_1^{nc}, \quad W_h = P_1^{nc}, \quad \Gamma_h = \mathscr{L}_{k-1}^0(\mathscr{T}_h), \tag{28}$$

and the degrees of freedom are shown in Fig. 2. For this element, optimal estimates

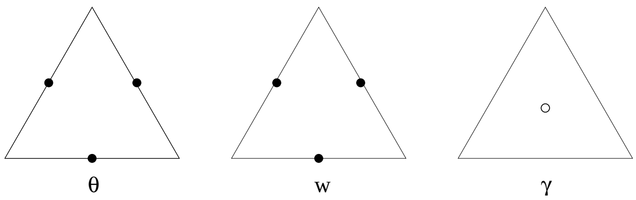

Fig. 2 D.o.f. for the nonconforming element

were proved in [13]

$$||\theta - \theta_h||_{1,h} + ||w - w_h||_{1,h} + ||\gamma - \gamma_h||_{\Gamma} + t||\gamma - \gamma_h||_0 \leq Ch||g||_0, \tag{29}$$

and in [10] for the L^2 error:

$$||\theta - \theta_h||_0 + ||w - w_h||_0 \leq Ch^2||g||_0. \tag{30}$$

3.2 DG-Elements without reduction operator P_h

Two families of elements for the formulation (20)-(21) were developed in [1]. In all the cases the transverse displacement w is approximated with piecewise polynomials of degree at most k, with $k \geq 2$, while the rotations θ and the shear stresses γ with piecewise polynomials of degree $\leq k-1$, and the methods differ in the amount of continuity required at the interelement boundaries. In all the cases the spaces satisfy

$$\nabla W_h \subseteq \Theta_h = \Gamma_h. \tag{31}$$

Example 3.2.1 In the first family of elements w is approximated by continuous finite elements, so that equations (20)-(21) simplify into:

$$a_h(\theta_h, \eta) + (\gamma_h, \nabla v - \eta) = (g, v), \qquad (\eta, v) \in \Theta_h \times W_h, \tag{32}$$

$$(\nabla w_h - \theta_h, \tau) - t^2(\gamma_h, \tau) = 0, \qquad \tau \in \Gamma_h. \tag{33}$$

Inclusion (31) forbids the use of a space Θ_h consisting of *continuous* functions. However, since w_h is continuous, it allows choices where the tangential component is continuous (as well as totally discontinuous choices). We recall here the choice that minimizes the number of degrees of freedom. For other possible choices see [1]. We take

$$W_h = \mathscr{L}_k^1, \quad \Theta_h = \Gamma_h = \mathbf{BDM}_{k-1}^R \qquad k \geq 2, \tag{34}$$

where $\mathbf{BDM}_{k-1}^R$ denotes the rotated Brezzi-Douglas-Marini space of degree $k-1$, i.e., the space of all piecewise polynomial vector fields of degree at most $k-1$ with tangential components continuous at the interelements [7]. With this choice, the inclusion (31) is clearly satisfied. The following estimates were proved in the norms (24):

$$|||\theta - \theta_h|||_\Theta + t\|\gamma - \gamma_h\|_0 \leq Ch^{k-1}(\|\theta\|_k + t\|\gamma\|_{k-1}), \tag{35}$$

$$\|\nabla(w - w_h)\|_0 \leq C(h^k + th^{k-1})(\|\theta\|_k + t\|\gamma\|_{k-1}), \tag{36}$$

and in L^2:

$$\|w - w_h\|_0 + \|\theta - \theta_h\|_0 \leq Ch^k(\|\theta\|_k + t\|\gamma\|_{k-1}). \tag{37}$$

Estimates (35)-(36) are optimal with respect to order of convergence (and also with respect to regularity for the case $k = 2$, according to (10)) while (37) is optimal for θ and suboptimal of one order for w.

Fig. 3 shows the degrees of freedom for the lowest order element of the family:

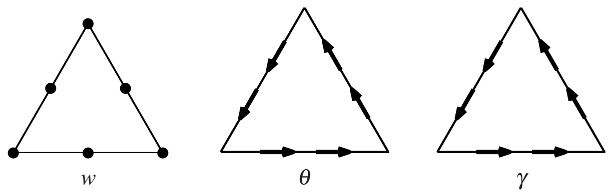

Fig. 3 Continuous w: lowest-order elements without reduction operator

Example 3.2.2 The second family consists of totally discontinuous elements. Thus, the spaces are

$$W_h = \mathscr{L}_k^0, \quad \Theta_h = \Gamma_h = \mathscr{L}_{k-1}^0 \qquad k \geq 2, \tag{38}$$

and the inclusion (31) is obviously verified. For this family the following error estimates were proved in the norms (24):

$$|||\theta - \theta_h|||_\Theta + t\|\gamma - \gamma_h\|_0 + [p_W(w - w_h, w - w_h)]^{1/2} \leq Ch^{k-1}(\|\theta\|_k + \|\gamma\|_{k-1}), \tag{39}$$

$$|||w - w_h|||_W \leq Ch^{k-1}(\|\theta\|_k + \|\gamma\|_{k-1} + \|w\|_k), \tag{40}$$

and in L^2:

$$\|\theta - \theta_h\|_0 + \|w - w_h\|_0 \le Ch^k(\|\theta\|_k + \|\gamma\|_{k-1}). \tag{41}$$

The bad feature of these estimates is the lack of the factor t in the norm $\|\gamma\|_{k-1}$ on the right hand side. Since this norm behaves like $t^{-(k-3/2)}$ as $t \to 0$, the extra factor of t helps to control the size of this term, and for $k = 2$ guarantees that it remains bounded. A better estimate in this respect can be obtained by assuming that the Helmholtz decomposition for γ holds. In this case we have:

$$\begin{aligned} \|\theta - \theta_h\|_\Theta + t\|\gamma - \gamma_h\|_0 + [p_W(w - w_h, w - w_h)]^{1/2} \\ \le Ch^{k-1}(\|\theta\|_k + t\|\gamma\|_{k-1} + \|\gamma\|_{H^{k-2}(\mathrm{div})}), \\ \|w - w_h\|_W \le Ch^{k-1}(\|\theta\|_k + t\|\gamma\|_{k-1} + \|\gamma\|_{H^{k-2}(\mathrm{div})} + \|w\|_k), \end{aligned} \tag{42}$$

and in L^2:

$$\|\theta - \theta_h\|_0 + \|w - w_h\|_0 \le Ch^k(\|\theta\|_k + t\|\gamma\|_{k-1} + \|\gamma\|_{H^{k-2}(\mathrm{div})}). \tag{43}$$

We point out that the regularity of γ is such that, for the lowest-order case $k = 2$, the Helmholtz decomposition holds, and estimates (42)-(43) are optimal with respect to regularity. Indeed, $\|\gamma\|_{H^{k-2}(\mathrm{div})} \equiv \|\operatorname{div}\gamma\|_0 \equiv \|g\|_0$ which does not explode when $t \to 0$. In terms of order of convergence they are optimal for θ, and suboptimal of one order for w. The lowest-order elements are depicted in Fig. 4.

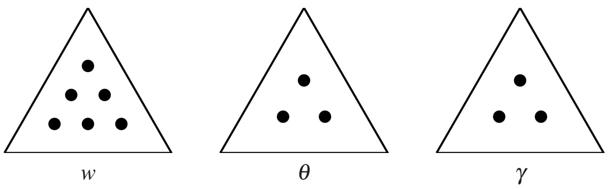

Fig. 4 Totally discontinuous elements without reduction operator: lowest-order elements

4 Conclusions

We presented a quick overview of some locking-free finite elements for Reissner-Mindlin plates, obtained through the use of Discontinuous Galerkin techniques. Since DG solutions are not required to satisfy the interelement continuity conditions of conforming finite elements, DG methods result more flexible and offer possibilities, in terms of degree of the finite elements, which are forbidden with conforming elements. For instance, the simple linear element of Example 3.1.1 would be unthinkable for conforming approximations. Similarly, the nonconforming linear element of Example 3.1.2 would have been hard to derive without the DG techniques.

The counterpart is that discontinuity implies an increasing of the number of unknowns, and efficient techniques to handle the final linear systems might be needed. The low order elements of Subsection 3.1 are very appealing, but their behavior might depends on the choice of the parameters in the penalty terms. By increasing these parameters one increases continuity, and the elements get closer to conforming elements, with the risk of locking. This dependence should be checked in practice, and sound numerical tests should be performed to compare the new elements with existing conforming elements, but this goes beyond the scope of this paper. We refer to [10] for numerical results on various linear nonconforming elements.

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