DISCONTINUOUS GALERKIN METHODS FOR ADVECTION-DIFFUSION-REACTION PROBLEMS

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Abstract. We apply the weighted-residual approach recently introduced in [7] to derive discontinuous Galerkin formulations for advection-diffusion-reaction problems. We devise the basic ingredients to ensure stability and optimal error estimates in suitable norms, and propose two new methods.

Key words. discontinuous Galerkin, advection-diffusion-reaction, inf-sup condition

AMS subject classifications. 65N30, 65N12, 65G99, 76R99

1. Introduction. In recent years Discontinuous Galerkin methods have become increasingly popular, and they have been used and analyzed for various kinds of applications: see, e.g., [2] for second order elliptic problems, [4]-[3] for Reissner-Mindlin plates and, for advection-diffusion problems, [11], [12], [17], [32], [5], [15],[18] and [9].

Most DG-methods for advection-diffusion or hyperbolic problems are constructed by specifying the numerical fluxes at the inter-elements and, as far as we know, the advection field is always assumed to be either constant or divergence free. In the present paper we follow a different path. From the one hand, we derive DG-formulations by applying the so called "weighted-residual" approach of [7]. In this approach a DG-method is written first in strong form, as a system of equations including the original PDE equation inside each element plus the necessary continuity conditions at interfaces. The variational form is then obtained by combining all these equations. In this way, the DG-method establishes a linear relationship between the residual inside each element and the jumps across inter-element boundaries. Such a linear relation permits to recover DG-methods proposed earlier in literature, and at the same time provides a framework for devising new DG-methods with the desired stability and consistency properties. As we shall show, this is possible, since stability and consistency can be ensured through a proper selection of the weights in the linear relationship, which in turn determines the DG-method.

On the other hand, and this is, in our opinion, the novelty of the present paper, we deal with a variable advection field which is not divergence-free. This, together with the presence of a variable reaction, makes the analysis more complicated than usually, surely more complicated than one could expect at first sight.

To ease the presentation we apply the "weighted-residual" approach to derive two DG-methods proposed in literature: the method introduced in [17], and that proposed in [18] and further analyzed in [9]. The former uses the non-symmetric NIPG method for the diffusion terms and upwind for the convective part of the flux. In the latter the diffusion terms are treated with three different DG-methods, and the whole physical flux is upwinded. This makes the approach well suited for strongly advection dominated problems (actually, the most interesting cases), but less adequate in the diffusion dominated or intermediate regimes. We also introduce two new methods.

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One of them, that we refer to as *minimal choice*, contains the minimum number of terms needed to get stability and optimal order of convergence in all regimes. The other one is a more refined method, that contains as a particular case the method [15] and the *minimal choice*.
Our formulation allows also to recover easily, for each of the methods analyzed, the corresponding SUPG-stabilized version. Many others methods could have been considered, but this would have made the paper practically unreadable. Moreover, our aim was not to compare the behavior of different schemes, but mostly to explore the possibilities and the ductility of the weighted residual approach for designing and analyzing DG-methods.

It is worth noticing that this approach seems to be particularly suited for understanding in a natural way which are the stabilization mechanisms, hidden in each DG method, responsible of the behavior of the DG approximation in the different regimes of the problem. It also provides a way to perform stability and a-priori error analysis in a unified framework. Furthermore, we think that it could be useful also for applications to a-posteriori error analysis, a field which is well developed for conforming approximations but much less studied for Discontinuous Galerkin approximations or even Stabilized methods. This surely deserves some further and future research.

Throughout the paper we shall use standard notations for norms and seminorms in Sobolev spaces. To keep homogeneity of dimensions, we recall that on a domain Ω of diameter L we define:

$$\|v\|^2_{k,\Omega} := \sum_{s=0}^{k} L^{2s}|v|^2_{s,\Omega} \qquad v \in H^k(\Omega) \quad k \geq 0, \tag{1.1}$$

$$\|v\|_{k,\infty,\Omega} := \sum_{s=0}^{k} L^{s}|v|_{s,\infty,\Omega} \qquad v \in W^{k,\infty}(\Omega) \quad k \geq 0. \tag{1.2}$$

The outline of the paper is as follows. In Section 2 we present the problem with all the assumptions necessary to the analysis, and we apply the "weighted residual" approach. In Section 3 we show examples of choices of the "weights", leading to four methods: the methods of [17] and [18], and two new methods. In Section 4 we deal with the approximation, and prove stability in a suitable DG-norm. We also prove stability in a norm of *SUPG*-type, thus providing control on the streamline derivative. Section 5 is devoted to a-priori error analysis, and optimal convergence is proved in both norms. Finally, in Section 6 we present an extensive set of numerical experiments to compare the methods and to validate our theoretical results.

2. Setting of the Problem. To ease the presentation we shall restrict ourselves to the two dimensional case, although the results here presented also hold in three dimensions. Let Ω be a bounded, convex, polygonal domain in $\mathbf{R}^2$, and let $\boldsymbol{\beta} = (\beta_1, \beta_2)^{\mathrm{T}}$ be the velocity vector field defined on $\overline{\Omega}$ with $\beta_i \in W^{1,\infty}(\Omega)$, $i = 1, 2$, $\gamma \in L^\infty(\Omega)$ the reaction coefficient, and ε a positive, constant, diffusivity coefficient. We define the *inflow* and *outflow* parts of $\Gamma = \partial\Omega$ in the usual fashion:

$$\begin{aligned} \Gamma^- &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) < 0\} = \text{inflow}, \\ \Gamma^+ &= \{x \in \Gamma : \ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) \geq 0\} = \text{outflow}, \end{aligned}$$

where $\mathbf{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$. Let Γ_D and Γ_N be the parts of the boundary Γ where Dirichlet and Neumann boundary conditions are assigned, so that $\Gamma = \overline{\Gamma_D \cup \Gamma_N}$, $\Gamma_D \cap \Gamma_N = \emptyset$. Thus,

$$\Gamma_D{}^{\pm} = \Gamma_D \cap \Gamma^{\pm}, \quad \Gamma_N{}^{\pm} = \Gamma_N \cap \Gamma^{\pm}.$$

Let $f \in L^2(\Omega)$, $g_D \in H^{3/2}(\Gamma_D)$, $g_N \in H^{1/2}(\Gamma_N)$. Consider the advection-diffusion-reaction problem

$$\begin{aligned} \mathrm{div}\boldsymbol{\sigma}(u) + \gamma u &= f && \text{in } \Omega, \\ u &= g_D && \text{on } \Gamma_D, \\ (\boldsymbol{\beta} u \chi_{\Gamma_N^-} - \varepsilon \boldsymbol{\nabla} u) \cdot \mathbf{n} &= g_N && \text{on } \Gamma_N, \end{aligned} \tag{2.1}$$

where $\boldsymbol{\sigma}(u)$ is the (physical) flux, given by

$$\boldsymbol{\sigma}(u) = -\varepsilon \boldsymbol{\nabla} u + \boldsymbol{\beta} u,$$

and $\chi_{\Gamma_N^-}$ is the characteristic function of Γ_N^-. The meaning of the boundary conditions on Γ_N is that the total flux is imposed on Γ_N^- while on Γ_N^+ only the diffusive flux is specified (see [18]).

Since the first equation in (2.1) is equivalent to $-\varepsilon\Delta u + \boldsymbol{\beta}\cdot\boldsymbol{\nabla} u + (\mathrm{div}\boldsymbol{\beta} + \gamma)u = f$, we introduce the "effective" reaction function $\varrho(x)$ and we make the assumption

$$\varrho(x) := \gamma(x) + \frac{1}{2}\mathrm{div}\boldsymbol{\beta}(x) \geq \varrho_0 \geq 0 \qquad \text{for all } x \in \Omega. \tag{2.2}$$

For the subsequent stability and error analysis we shall make the following assumptions on the coefficients: the advective field has neither closed curves nor stationary points, i.e.,

$$\boldsymbol{\beta} \text{ has no closed curves} \qquad \text{and} \qquad |\boldsymbol{\beta}(x)| \neq 0 \qquad \text{for all } x \in \Omega. \tag{2.3}$$

This implies, as we shall see later on (see Remark 2.1 below and Appendix A), that

$$\exists\, \eta \in W^{k+1,\infty}(\Omega) \qquad \text{such that} \qquad \boldsymbol{\beta}\cdot\boldsymbol{\nabla}\eta \geq 2b_0 := 2\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L} \quad \text{in } \Omega. \tag{\textbf{H1}}$$

Furthermore, we assume that:

$$\exists\, c_\beta > 0 \text{ such that } |\boldsymbol{\beta}(x)| \geq c_\beta \|\boldsymbol{\beta}\|_{1,\infty,\Omega} \qquad \forall x \in \Omega, \tag{\textbf{H2}}$$

and, given a shape-regular family $\mathcal{T}_h$ of decompositions of Ω into triangles T:

$$\exists\, c_\varrho > 0 \text{ such that } \forall T \in \mathcal{T}_h : \quad \|\varrho\|_{0,\infty,T} \leq c_\varrho(\min_T \varrho(x) + b_0). \tag{\textbf{H3}}$$

REMARK 2.1. *Assumption* (2.3)*, together with the regularity* $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$*, ensures the well-posedness of the continuous problem in the pure hyperbolic limit (*$\varepsilon = 0$*). (See [14] and also [27] for details). Condition* **(H1)** *is based on a result first established in [14, Lemma 2.3] under more regularity assumptions on* $\boldsymbol{\beta}$*. Namely, for* $\boldsymbol{\beta} \in \mathcal{C}^k(\mathcal{U}), k \geq 1$ *satisfying* (2.3)*,* $\mathcal{U}$ *being some neighborhood of* $\overline{\Omega}$*, the authors show the existence of* $\eta \in \mathcal{C}^k(\mathcal{U})$ *verifying* $\boldsymbol{\beta}\cdot\nabla\eta \geq b_0 > 0$ *in* Ω*. However, by revising the proof in [14], it can be seen that the result holds true also if* $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$*, provided it satisfies* (2.3)*(see Appendix A for details).*

Assumption **(H2)** *excludes undesirable situations of a small but highly oscillatory advection field, and provides useful relations among norms. Indeed, from* (1.2) *we deduce*

$$\begin{aligned} c_\beta \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L} &\leq b_0 := \frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L} \leq \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L}, \\ |\boldsymbol{\beta}|_{1,\infty,\Omega} &\leq \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L} \leq \frac{1}{c_\beta}\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L} = \frac{b_0}{c_\beta}. \end{aligned} \tag{2.4}$$

Hypothesis **(H3)** *is always verified in the advection dominated regime (it says nothing more than $\varrho \in L^\infty(\Omega)$). Instead, when the advection field is negligible, it forbids the problem to shift from reaction-dominated to diffusion-dominated within a single element. Note that, since we are interested in the case where the diffusion coefficient ε is very small, what we refer to as "diffusion-dominated" problem (that is, when both reaction and advection are also very small) has little practical interest.*

Let again $\mathcal{T}_h$ be a shape-regular family of decompositions of Ω into triangles T, such that each (open) boundary edge belongs either to Γ_D, or to Γ_N^+ or to Γ_N^- (in other words, we avoid edges that belong to two different types of boundary). We denote by h_T the diameter of T, and we set $h = \max_{T\in\mathcal{T}_h} h_T$. Since we look for a solution of (2.1) a-priori discontinuous, we need to recall the definition of typical tools such as *averages* and *jumps* on the edges for scalar and for vector-valued functions. Let T_1 and T_2 be two neighboring elements, let $\mathbf{n}^1$ and $\mathbf{n}^2$ be their outward normal unit vectors, and let φ^i and $\boldsymbol{\tau}^i$ be the restriction of φ and $\boldsymbol{\tau}$ to T_i, $(i = 1, 2)$, respectively. Following [2] we set:

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\, \varphi \,]\!] = \varphi^1\mathbf{n}^1 + \varphi^2\mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ, \tag{2.5}$$

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \quad [\![\, \boldsymbol{\tau} \,]\!] = \boldsymbol{\tau}^1\cdot\mathbf{n}^1 + \boldsymbol{\tau}^2\cdot\mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ, \tag{2.6}$$

where $\mathcal{E}_h^\circ$ is the set of interior edges e. For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we set

$$[\![\, \varphi \,]\!] = \varphi\mathbf{n}, \quad \{\varphi\} = \varphi, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}. \tag{2.7}$$

For future purposes we also introduce a weighted average, for both scalar and vector-valued functions, as follows. With each internal edge e, shared by elements T_1 and T_2, we associate two real nonnegative numbers α^1 and α^2, with $\alpha^1 + \alpha^2 = 1$, and we define

$$\{\boldsymbol{\tau}\}_\alpha = \alpha^1\boldsymbol{\tau}^1 + \alpha^2\boldsymbol{\tau}^2 \quad \text{on internal edges.} \tag{2.8}$$

As shown for instance in [8] for a pure hyperbolic problem, a proper choice of α^1 and α^2 will introduce a stabilizing effect of upwind type into the scheme. We note that the arithmetic average is obtained for $\alpha^1 = \alpha^2 = 1/2$, while the classical upwind flux is obtained when $\alpha^i = (\mathrm{sign}(\boldsymbol{\beta}\cdot\mathbf{n}^i)+1)/2$ for $i = 1, 2$ (where, as usual, $\mathrm{sign}(x) = x/|x|$ for $x \neq 0$ and $\mathrm{sign}(0) = 0$). Indeed, the following relation holds:

$$\{\boldsymbol{\tau}\}_\alpha = \{\boldsymbol{\tau}\} + \frac{[\![\, \alpha \,]\!]}{2}[\![\, \boldsymbol{\tau} \,]\!], \tag{2.9}$$

so that, if for instance T_1 is the upwind triangle, i.e. $\boldsymbol{\beta}\cdot\mathbf{n}^1 > 0$, then $\alpha = (1, 0)$ and

$$\{\boldsymbol{\tau}\}_{upw} = \{\boldsymbol{\tau}\} + \frac{\mathbf{n}^1}{2}[\![\, \boldsymbol{\tau} \,]\!] = \boldsymbol{\tau}^1 \qquad \{\boldsymbol{\tau}\}_{dw} = \{\boldsymbol{\tau}\} + \frac{\mathbf{n}^2}{2}[\![\, \boldsymbol{\tau} \,]\!] = \boldsymbol{\tau}^2. \tag{2.10}$$

Taking $\alpha^i = 1/2 + t\,\mathrm{sign}(\boldsymbol{\beta}\cdot\mathbf{n}^i)$ ($i = 1, 2$) will allow, choosing t with $0 < t_0 \le t \le 1/2$ on each edge, to tune up the quantity of upwind.

We shall make extensive use of the following identity [2, formula (3.3)]:

$$\sum_{T\in\mathcal{T}_h}\int_{\partial T}\boldsymbol{\tau}\cdot\mathbf{n}\varphi = \sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\tau}\}\cdot[\![\, \varphi \,]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e[\![\, \boldsymbol{\tau} \,]\!]\{\varphi\}, \tag{2.11}$$

of the trace inequality ([1], [2]):

$$\|w\|_{0,e}^2 \le C_t^2(|e|^{-1}\|w\|_{0,T}^2 + |e||w|_{1,T}^2), \qquad e \subset \partial T,\ w \in H^1(T), \tag{2.12}$$

with C_t a constant only depending on the minimum angle of T, and $|e|$ = length of the edge e, and finally of the DG-Poincaré inequality [6]:

$$\|v\|_{0,\Omega} \le L\, C_P\Big(|v|_{1,h}^2 + \sum_{e\notin\Gamma_N} \frac{1}{|e|}\|[\![v]\!]\|_{0,e}^2\Big)^{1/2}. \tag{2.13}$$

With the previous definitions, problem (2.1) is equivalent to

$$\begin{cases} \mathrm{div}\boldsymbol{\sigma}(u) + \gamma u & = f \quad \text{in each } T \in \mathcal{T}_h, \\ [\![\boldsymbol{\sigma}(u)]\!] & = 0 \quad \text{on each } e \in \mathcal{E}_h^\circ, \\ [\![u]\!] & = 0 \quad \text{on each } e \in \mathcal{E}_h^\circ, \\ u & = g_D \quad \text{on each } e \in \Gamma_D, \\ (\boldsymbol{\beta} u \chi_{\Gamma_N^-} - \varepsilon \boldsymbol{\nabla} u)\cdot \mathbf{n} & = g_N \quad \text{on each } e \in \Gamma_N. \end{cases} \tag{2.14}$$

Following the approach of [7], we shall introduce a variational formulation of (2.14) in which each of the equations above has the same relevance, and is therefore treated in the same fashion. To do so, we introduce the space

$$V(\mathcal{T}_h) := \{v \in L^2(\Omega) \text{ such that } v_{|T} \in H^s(T)\ \forall T \in \mathcal{T}_h, \ \ s > 3/2\},$$

and we assume that we have five operators $\mathcal{B}_0,\ \mathcal{B}_1,\ \mathcal{B}_2,\ \mathcal{B}_1^D,\ \mathcal{B}_2^N$ from $V(\mathcal{T}_h)$ to $L^2(\Omega),\ \mathbf{L}^2(\mathcal{E}_h^\circ),\ L^2(\mathcal{E}_h^\circ),\ L^2(\Gamma_D),\ L^2(\Gamma_N)$, respectively. Then we consider the problem

$$\begin{cases} \text{Find } u \in V(\mathcal{T}_h) \text{ such that } \forall v \in V(\mathcal{T}_h) \\ \displaystyle\int_\Omega (\mathrm{div}_h \boldsymbol{\sigma}(u) + \gamma u - f)\mathcal{B}_0 v + \sum_{e\in\mathcal{E}_h^\circ}\int_e [\![u]\!]\cdot \mathcal{B}_1 v + \sum_{e\in\mathcal{E}_h^\circ}\int_e [\![\boldsymbol{\sigma}(u)]\!] \mathcal{B}_2 v \\ \displaystyle + \sum_{e\in\Gamma_D}\int_e (u - g_D)\,\mathcal{B}_1^D v + \sum_{e\in\Gamma_N}\int_e (\boldsymbol{\beta} u\chi_{\Gamma_N^-} - \varepsilon\boldsymbol{\nabla} u)\cdot\mathbf{n} - g_N)\,\mathcal{B}_2^N v = 0, \end{cases} \tag{2.15}$$

where div_h denotes the divergence element by element.

Different choices of the $\mathcal{B}'$s operators will give rise to different formulations. Since the solution of the original problem (2.1) is always a solution of (2.15), if we ensure uniqueness of the solution of (2.15), such a solution will coincide with the solution of the original problem. Sufficient conditions on the operators $\mathcal{B}$ to guarantee uniqueness of the solution of (2.15) are given in [7, Theorem 1]. In the next section we shall present some choices of the operators verifying the hypotheses of the cited theorem.

3. Variational formulations. We will present four examples of different choices for the operators in (2.15). Two of them reproduce known formulations, while the other two will give rise to new methods.

Example 1 We set

$$\begin{aligned} &\mathcal{B}_0 v_{|T} = v,\ \forall T \in \mathcal{T}_h, \qquad \mathcal{B}_1 v_{|e} = c_e\,\frac{\varepsilon}{|e|}[\![v]\!] + \frac{\mathbf{n}^+}{2}[\![\boldsymbol{\beta} v]\!],\ \forall e \in \mathcal{E}_h^\circ, \\ &\mathcal{B}_2 v_{|e} = -\{v\},\ \forall e \in \mathcal{E}_h^\circ\ , \\ &\mathcal{B}_1^D v_{|e} = c_e\,\frac{\varepsilon}{|e|}[\![v]\!]\cdot\mathbf{n} - \boldsymbol{\beta}\cdot\mathbf{n} v,\ \forall e \in \Gamma_D^-, \qquad \mathcal{B}_2^N v_{|e} = -v,\ \forall e \in \Gamma_N^-. \end{aligned} \tag{3.1}$$

In (3.1) $\mathbf{n}^+$ is the normal to e such that $\boldsymbol{\beta}\cdot\mathbf{n}^+ \geq 0$, and c_e is a positive constant such that (see [2])

$$c_e \geq \eta_0 > 0 \quad \forall e \in \mathcal{E}_h\ . \tag{3.2}$$

We shall see that the definition of the operators on Γ^+ can be made arbitrary, without compromising the stability or consistency properties of the resulting methods. We can choose, for instance,

$$\mathcal{B}_1^D v = c_e\,\frac{\varepsilon}{|e|}v \quad \text{on} \quad e\in\Gamma_D^+, \qquad \mathcal{B}_2^N v = -v \quad \text{on} \quad \Gamma_N^+.$$

With these choices, and setting

$$S_e = c_e\,\frac{\varepsilon}{|e|},$$

problem (2.15) reads:

$$\begin{aligned}
0 = &\sum_{T\in\mathcal{T}_h}\int_T(\mathrm{div}\boldsymbol{\sigma}(u)+\gamma u - f)v + \sum_{e\in\mathcal{E}_h^\circ}\int_e[\![\, u\,]\!]\cdot(S_e[\![\, v\,]\!] + \frac{\mathbf{n}^+}{2}[\![\,\boldsymbol{\beta}v\,]\!])\\
&-\sum_{e\in\mathcal{E}_h^\circ}\int_e[\![\,\boldsymbol{\sigma}(u)\,]\!]\{v\} + \sum_{e\in\Gamma_D^-}\int_e(u-g_D)\cdot(S_e[\![\, v\,]\!]-\boldsymbol{\beta}v)\cdot\mathbf{n}\\
&+\sum_{e\in\Gamma_D^+}S_e\int_e(u-g_D)\,v - \sum_{e\in\Gamma_N}\int_e((\boldsymbol{\beta}u\chi_{\Gamma_N^-}-\varepsilon\boldsymbol{\nabla}_h u)\cdot\mathbf{n}-g_N)v.
\end{aligned}\tag{3.3}$$

Using the identity (2.11) we have

$$\int_\Omega \mathrm{div}_h\boldsymbol{\sigma}(u)\,v = -\int_\Omega\boldsymbol{\sigma}(u)\cdot\boldsymbol{\nabla}_h v + \sum_{e\in\mathcal{E}_h^\circ}\int_e[\![\,\boldsymbol{\sigma}(u)\,]\!]\{v\} + \sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\sigma}(u)\}\cdot[\![\, v\,]\!].$$

Substituting in (3.3), taking into account the continuity of $\boldsymbol{\beta}$, and using (2.10) we obtain the following formulation:

$$\left\{\begin{aligned}
&\text{Find } u\in V(\mathcal{T}_h) \text{ such that } \forall v\in V(\mathcal{T}_h)\\
&\int_\Omega(\gamma uv-\boldsymbol{\sigma}(u)\cdot\boldsymbol{\nabla}_h v) + \sum_{e\notin\Gamma_N}S_e\int_e[\![\, u\,]\!]\cdot[\![\, v\,]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e\{\boldsymbol{\beta}u\}_{upw}\cdot[\![\, v\,]\!]\\
&-\sum_{e\notin\Gamma_N}\int_e\{\varepsilon\boldsymbol{\nabla}_h u\}\cdot[\![\, v\,]\!] + \int_{\Gamma^+}\boldsymbol{\beta}\cdot\mathbf{n}uv\\
&=\sum_{T\in\mathcal{T}_h}\int_T f\,v + \sum_{e\in\Gamma_D}S_e\int_e g_D\,v - \sum_{e\in\Gamma_D^-}\int_e\boldsymbol{\beta}\cdot\mathbf{n}g_Dv - \sum_{e\in\Gamma_N}\int_e g_Nv.
\end{aligned}\right.\tag{3.4}$$

We observe that for the diffusive part this method gives the so-called incomplete interior penalty (IIPG) proposed and analyzed in [30], while the advective part is upwinded through the operator $\mathcal{B}_1$.

Example 2 We set

$$\begin{aligned}
&\mathcal{B}_0 v_{|T} = v,\ \forall T\in\mathcal{T}_h,\\
&\mathcal{B}_1 v_{|e} = c_e\frac{\varepsilon}{|e|}[\![\, v\,]\!] + \{\varepsilon\boldsymbol{\nabla}_h v\} + \frac{\mathbf{n}^+}{2}[\![\,\boldsymbol{\beta}v\,]\!],\ \forall e\in\mathcal{E}_h^\circ,\\
&\mathcal{B}_2 v_{|e} = -\{v\},\ \forall e\in\mathcal{E}_h^\circ, \qquad \mathcal{B}_2^N v_{|e} = -v\ \forall e\in\Gamma_N,\\
&\mathcal{B}_1^D v_{|e} = c_e\frac{\varepsilon}{|e|}v + (\varepsilon\boldsymbol{\nabla}_h v - \boldsymbol{\beta}v\chi_{\Gamma_D^-})\cdot\mathbf{n},\ \forall e\in\Gamma_D.
\end{aligned}\tag{3.5}$$

These choices reproduce the method introduced in [17] for the case $\gamma = 0$ and different boundary conditions. Indeed, in [17] the flux was not assigned at the inflow, and the boundary conditions were, with our notation,

$$u = g_D \quad \text{on } \Gamma_D \equiv \Gamma \setminus \Gamma_N^+, \qquad (-\varepsilon \boldsymbol{\nabla} u)\cdot \mathbf{n} = g_N \quad \text{on } \Gamma_N^+\,, \quad \Gamma_N^- = \emptyset.$$

In (3.5) the diffusive part corresponds to the NIPG method of [26], and the advective part is upwinded through $\mathcal{B}_1$. Substituting (3.5) in (2.15), and using (2.10) and the continuity of $\boldsymbol{\beta}$ leads to the problem:

$$\begin{cases} \text{Find} \quad u \in V(\mathcal{T}_h) \quad \text{such that} \quad \forall\, v \in V(\mathcal{T}_h) \\ \displaystyle\int_\Omega (\gamma u v - \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v) + \sum_{e\notin\Gamma_N} S_e \int_e [\![\, u \,]\!]\cdot[\![\, v \,]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e \{\boldsymbol{\beta} u\}_{upw}\cdot[\![\, v \,]\!] \\ \displaystyle - \sum_{e\notin\Gamma_N}\int_e (\{\varepsilon\boldsymbol{\nabla}_h u\}\cdot[\![\, v \,]\!] - [\![\, u \,]\!]\cdot\{\varepsilon\boldsymbol{\nabla}_h v\}) + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n} u v \\ \displaystyle = \sum_{T\in\mathcal{T}_h}\int_T f\, v + \sum_{e\in\Gamma_D}\int_e g_D(S_e v + (\varepsilon\boldsymbol{\nabla}_h v - \boldsymbol{\beta} v \chi_{\Gamma_D^-})\cdot\mathbf{n}) - \sum_{e\in\Gamma_N}\int_e g_N v. \end{cases} \tag{3.6}$$

Example 3 We set

$$\begin{aligned} &\mathcal{B}_0 v_{|T} = v, \ \forall T\in\mathcal{T}_h, \\ &\mathcal{B}_1 v_{|e} = c_e\frac{\varepsilon}{h}[\![\, v \,]\!] - \theta\{\varepsilon\boldsymbol{\nabla} v\}_{upw}, \ \forall e\in\mathcal{E}_h^\circ, \\ &\mathcal{B}_2 v_{|e} = -\{v\}_{dw}, \ \forall e\in\mathcal{E}_h^\circ, \qquad \mathcal{B}_2^N v_{|e} = -v, \ \forall e\in\Gamma_N, \\ &\mathcal{B}_1^D v_{|e} = c_e\frac{\varepsilon}{h} v - (\theta\varepsilon\boldsymbol{\nabla} v + \boldsymbol{\beta} v\chi_{\Gamma_D^-})\cdot\mathbf{n}, \ \forall e\in\Gamma_D^-, \end{aligned}$$

where θ is a parameter that allows to include various formulations for treating the diffusive part: symmetric for $\theta = 1$, skew-symmetric for $\theta = -1$, and neutral for $\theta = 0$. This choice of the operators corresponds to the method introduced in [18] and analyzed in [9]. By substituting in (2.15), integrating by parts and rearranging terms we obtain the following scheme:

$$\begin{cases} \text{Find } u \in V(\mathcal{T}_h) \text{ such that } \forall v \in V(\mathcal{T}_h) \\ \displaystyle\int_\Omega (\gamma u v - \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v) + \sum_{e\notin\Gamma_N} S_e \int_e [\![\, u \,]\!]\cdot[\![\, v \,]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e \{\boldsymbol{\beta} u\}_{upw}\cdot[\![\, v \,]\!] \\ \displaystyle - \sum_{e\in\mathcal{E}_h^\circ}\int_e (\{\varepsilon\boldsymbol{\nabla}_h u\}_{upw}\cdot[\![\, v \,]\!] + \theta[\![\, u \,]\!]\cdot\{\varepsilon\boldsymbol{\nabla}_h v\}_{upw}) + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n} u v \\ \displaystyle - \sum_{e\in\Gamma_D}\int_e (\varepsilon\boldsymbol{\nabla}_h u\cdot\mathbf{n} v + \theta u\varepsilon\boldsymbol{\nabla}_h v\cdot\mathbf{n}) \\ \displaystyle = \sum_{T\in\mathcal{T}_h}\int_T f v + \sum_{e\in\Gamma_D}\int_e g_D(S_e[\![\, v \,]\!] - \theta\varepsilon\boldsymbol{\nabla}_h v - \boldsymbol{\beta} v\chi_{\Gamma_D^-})\cdot\mathbf{n} - \sum_{e\in\Gamma_N}\int_e g_N v. \end{cases} \tag{3.7}$$

In (3.7) the whole flux $\boldsymbol{\sigma}(u)$ is upwinded through the operator $\mathcal{B}_2$, but the upwind effect for the advective part is exactly the same as in methods (3.4) and (3.6).

Example 4 Let $\{\cdot\}_\alpha$ be the weighted average defined in (2.8)-(2.9). We set

$$
\begin{aligned}
&\mathcal{B}_0 v_{|T} = v, \ \forall T \in \mathcal{T}_h \\
&\mathcal{B}_1 v_{|e} = c_e \frac{\varepsilon}{|e|} [\![v]\!] + \theta(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}), \ \forall e \in \mathcal{E}_h^\circ, \\
&\mathcal{B}_2 v_{|e} = -\{v\}_{1-\alpha}, \ \forall e \in \mathcal{E}_h^\circ, \qquad \mathcal{B}_2^N v_{|e} = -v \ \forall e \in \Gamma_N, \\
&\mathcal{B}_1^D v_{|e} = c_e \frac{\varepsilon}{h} v - (\theta \varepsilon \boldsymbol{\nabla}_h v + \boldsymbol{\beta} v \chi_{\Gamma_D^-}) \cdot \mathbf{n}, \ \forall e \in \Gamma_D.
\end{aligned}
$$

Substituting in (2.15) yields

$$
\left\{
\begin{aligned}
&\text{Find } u \in V(\mathcal{T}_h) \quad \text{such that} \quad \forall\, v \in V(\mathcal{T}_h) \\
&\int_\Omega \gamma u v - \boldsymbol{\sigma}(u) \cdot \boldsymbol{\nabla}_h v + \sum_{e \notin \Gamma_N} S_e \int_e [\![u]\!] \cdot [\![v]\!] - \theta \sum_{e \in \mathcal{E}_h^\circ} \int_e [\![u]\!] \cdot \{\boldsymbol{\beta} v\} \\
&+ \sum_{e \in \mathcal{E}_h^\circ} \int_e (\{\boldsymbol{\sigma}(u)\}_\alpha \cdot [\![v]\!] + \theta [\![u]\!] \cdot \{\boldsymbol{\sigma}(v)\}_\alpha) + \sum_{e \in \Gamma^+} \int_e \boldsymbol{\beta} \cdot \mathbf{n} u v \\
&- \sum_{e \in \Gamma_D} \int_e (\varepsilon \boldsymbol{\nabla}_h u \cdot \mathbf{n}\, v + \theta u\, \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n}) \\
&= \sum_{T \in \mathcal{T}_h} \int_T f v + \sum_{e \in \Gamma_D} g_D (S_e [\![v]\!] - \theta \varepsilon \boldsymbol{\nabla}_h v - \boldsymbol{\beta} v \chi_{\Gamma_D^-}) \cdot \mathbf{n} - \sum_{e \in \Gamma_N} \int_e g_N v.
\end{aligned}
\right. \tag{3.8}
$$

In (3.8) θ is again a parameter that allows to include different treatments of the diffusive part: symmetric for $\theta = 1$ SIPG(α)([29],[16]), non-symmetric for $\theta = -1$ and neutral for $\theta = 0$. However, as we shall see in Remark 4.2, the case $\theta = -1$ gives rise to a formulation which is stable in a norm too weak, with a consequent loss of accuracy in the error estimates. Thus, it will not be further considered. The upwind is achieved in (3.8) through both operators $\mathcal{B}_1$ and $\mathcal{B}_2$. Moreover, the use of the weighted average (2.8) should allow to tune the amount of upwind on each edge. As a consequence, the formulation enjoys the nice feature of adapting easily from the advection dominated to the diffusion dominated regime.
All the above formulations share the common form:

$$
\left\{
\begin{aligned}
&\text{Find } u \in V(\mathcal{T}_h) \text{ such that :} \\
&a_h(u, v) = L(v) \quad \forall v \in V(\mathcal{T}_h).
\end{aligned}
\right.
$$

REMARK 3.1. *In all cases, for obtaining the corresponding SUPG-stabilized DG formulations, one only needs to change the definition of the operator $\mathcal{B}_0$ into $\mathcal{B}_0 v = v + \mathfrak{c}_T \boldsymbol{\beta} \cdot \nabla v_h$ on each $T \in \mathcal{T}_h$, $\mathfrak{c}_T$ being a constant varying elementwise and depending on h_T and the coefficients of the problem $\boldsymbol{\beta}, \varepsilon, \gamma$ (see [22], [19] and [18]).*

4. Approximation. With any integer $k \geq 1$ we associate the finite element space of discontinuous piecewise polynomial functions

$$
V_h^k = \{v \in \mathrm{L}^2(\Omega) : \ v_{|T} \in \mathrm{P}^k(T) \quad \forall T \in \mathcal{T}_h\},
$$

where, as usual, $\mathrm{P}^k(T)$ is the space of polynomials of degree at most k on T. Replacing $V(\mathcal{T}_h)$ by V_h^k, we get the discrete problems, all sharing the form

$$
\left\{
\begin{aligned}
&\text{Find } u_h \in V_h^k \text{ such that :} \\
&a_h(u_h, v_h) = L(v_h) \quad \forall v_h \in V_h^k.
\end{aligned}
\right. \tag{4.1}
$$

Consistency. Consistency holds by construction in all the cases, so that

$$a_h(u-u_h, v_h) = 0 \qquad \forall v_h \in V_h^k. \tag{4.2}$$

Stability. We shall prove stability in the norm

$$|||v|||^2 = |||v|||_d^2 + |||v|||_{rc}^2, \tag{4.3}$$

with

$$|||v|||_d^2 := \varepsilon |v|_{1,h}^2 + \varepsilon \|v\|_j^2 := \varepsilon |v|_{1,h}^2 + \sum_{e \notin \Gamma_N} \frac{\varepsilon}{|e|} \|[\![v]\!]\|_{0,e}^2,$$
$$|||v|||_{rc}^2 := \|(\overline{\varrho}+b_0)^{1/2} v\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_h} \||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2} [\![v]\!]\|_{0,e}^2,$$

where $|\cdot|_{1,h}$ denotes the broken H^1-seminorm, $b_0 = \|\boldsymbol{\beta}\|_{0,\infty}/L$ is defined in **(H1)**, and $\overline{\varrho}$ is the piecewise constant function defined as

$$\overline{\varrho}(x)_{|T} = \overline{\varrho}_{|T}, \quad \overline{\varrho}_{|T} = \min_{x \in T} \varrho(x), \quad \forall T \in \mathcal{T}_h. \tag{4.4}$$

Analogously, it will be useful to write the bilinear forms as

$$a_h(u,v) = a_h^d(u,v) + a_h^{rc}(u,v). \tag{4.5}$$

For simplicity, we start by considering the method (3.4), which corresponds to the "minimal choice" for the operators. Then we have:

$$a_h^d(u,v) = \int_\Omega \varepsilon \boldsymbol{\nabla}_h u \cdot \boldsymbol{\nabla}_h v + \sum_{e \notin \Gamma_N} \int_e (S_e [\![u]\!] - \{\varepsilon \boldsymbol{\nabla}_h u\}) \cdot [\![v]\!], \tag{4.6}$$

$$a_h^{rc}(u,v) = \int_\Omega (\gamma uv - u\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v) + \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\beta} u\}_{upw} \cdot [\![v]\!] + \int_{\Gamma^+} \boldsymbol{\beta}\cdot\mathbf{n} uv. \tag{4.7}$$

We note that, using (2.12) and arguing as in [2] we can easily see that there exists a (geometric) constant C_g, depending only on the degree of the polynomials and on the minimum angle of the decomposition such that

$$\sum_{e \notin \Gamma_N} \int_e \left| \{\varepsilon \boldsymbol{\nabla}_h u\} [\![v]\!] \right| \le C_g \varepsilon |u|_{1,h} \|v\|_j \qquad \forall u \in V_h^k, \ \forall v \in V(\mathcal{T}_h). \tag{4.8}$$

This implies that there exists a constant $C_d > 0$ such that

$$a_h^d(u,v) \le C_d\, |||u|||_d |||v|||_d \qquad u \in V_h^k,\ v \in V(\mathcal{T}_h), \tag{4.9}$$

and, for η_0 in (3.2) verifying

$$\eta_0 > C_g^2/4 \tag{4.10}$$

there exists a positive constant α_d such that:

$$a_h^d(v,v) \ge \alpha_d |||v|||_d^2 \qquad v \in V_h^k. \tag{4.11}$$

We also note that, in general, one would rather require, say,

$$\eta_0 > \max\{C_g^2, 1\} \tag{4.12}$$

in order to have a quantifiable constant like $\alpha_d = 1/2$. In any case, the diffusive part alone would easily verify stability in all the methods. However, the technique of taking $v = u$, which is possibly the easiest way of proving stability, will not be sufficient when the reactive-advective part is also present, as it does not provide control on the L^2-norm when advection dominates. Indeed, in all the cases we would only have:

$$a_h^{rc}(v, v) \geq \|\overline{\varrho}^{1/2} v\|_{0,\Omega}^2 + \sum_{e\in\mathcal{E}_h} \||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![v]\!]\|_{0,e}^2 \qquad v \in V_h^k.$$

We will then prove stability in the norm (4.3) through an inf-sup condition. For that, following [22], we introduce the "weighting function" $\chi = exp(-\eta)$, with η defined in **(H1)**. The assumptions on η imply the existence of three positive constants $\chi_1^*, \chi_2^*, \chi_3^*$ such that

$$\chi_1^* \leq \chi \leq \chi_2^*, \qquad |\nabla\chi| \leq \chi_3^*. \tag{4.13}$$

Our "weighting function" will be slightly different. Indeed we shall take

$$\varphi = \chi + \kappa \tag{4.14}$$

where κ is a constant such that

$$\chi_1^* + \kappa > 6\, C_P\, L\chi_3^* \qquad \chi_1^* + \kappa > (\chi_2^* + \kappa)/2, \tag{4.15}$$

and C_P is the Poincaré constant appearing in (2.13).

The next Lemma is a generalization to the case of variable $\boldsymbol{\beta}$ of that given in [20] for pure hyperbolic problems. See also [22] for the equivalent result for SUPG-stabilized method, and [28] for the conforming Residual-Free Bubbles method. We point out however that here, thanks to the choice (4.14), we were able to remove the condition "ε sufficiently small".

LEMMA 4.1. *Let $a_h(\cdot,\cdot)$ be defined in* (4.5)-(4.7)*, with*

$$\eta_0 > \max\{9C_g^2/4, 1\}. \tag{4.16}$$

Then, for every κ satisfying (4.15)*, the corresponding φ defined in* (4.14) *verifies:*

$$a_h^d(v_h, \varphi v_h) \geq \frac{\chi_1^* + \kappa}{6} |||v_h|||_d^2 \tag{4.17}$$

$$a_h^{rc}(v_h, \varphi v_h) \geq \frac{\chi_1^*}{2} |||v_h|||_{rc}^2, \tag{4.18}$$

$$|||\varphi v_h||| \leq \frac{\sqrt{145}}{6}(\chi_1^* + \kappa)|||v_h|||. \tag{4.19}$$

Proof. To simplify the notation we shall write

$$\alpha_1 = \chi_1^* + \kappa, \quad \alpha_2 = \chi_2^* + \kappa, \quad \alpha_3 \equiv \chi_3^*$$

so that

$$\alpha_1 \le \varphi \le \alpha_2, \quad |\nabla\varphi| \le \alpha_3, \tag{4.20}$$

$$i) \quad \alpha_1 > 6\, C_P\, L\, \alpha_3, \qquad ii) \quad 2\alpha_1 > \alpha_2. \tag{4.21}$$

Conditions (4.8), and (4.20) give

$$\begin{aligned} a_h^d(v_h, \varphi v_h) &= \int_\Omega \varepsilon|\boldsymbol{\nabla}_h v_h|^2\varphi + \sum_{e\notin\Gamma_N}\int_e (S_e[\![v_h]\!] - \{\varepsilon\boldsymbol{\nabla}_h v_h\})\cdot[\![v]\!]\varphi + \int_\Omega \varepsilon\boldsymbol{\nabla}_h v_h\cdot\boldsymbol{\nabla}\varphi v_h \\ &\ge \varepsilon\Big(\alpha_1(|v_h|^2_{1,h} + \eta_0\|v_h\|_j^2) - \alpha_2\, C_g|v_h|_{1,h}\,\|v_h\|_j - \alpha_3|v_h|_{1,h}\|v_h\|_{0,\Omega}\Big). \end{aligned}$$

This, using (4.21; ii)) and (4.16), then $\eta_0 \ge 1$ and (2.13), and finally (4.21; i)), gives easily

$$\begin{aligned} a_h^d(v_h, \varphi v_h) &\ge \varepsilon\Big(\frac{\alpha_1}{3}(|v_h|^2_{1,h} + \eta_0\|v_h\|_j^2) - \alpha_3|v_h|_{1,h}\|v_h\|_{0,\Omega}\Big) \\ &\ge \varepsilon\frac{\alpha_1}{3}\Big(|v_h|^2_{1,h} + \|v_h\|_j^2\Big) - \alpha_3\, C_P\, L\,|||v_h|||_d^2 \ge \frac{\alpha_1}{6}|||v_h|||_d^2\,, \end{aligned}$$

that is (4.17). As regards the reactive-convective part, we observe that, after integration by parts, using (2.11) and the continuity of $\boldsymbol{\beta}$ and φ we get:

$$\begin{aligned} -\int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h(\varphi v_h)v_h &= -\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\varphi)v_h^2 - \frac{1}{2}\int_\Omega \boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h(v_h^2)\varphi \\ &= -\frac{1}{2}\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\varphi)v_h^2 + \frac{1}{2}\int_\Omega(\mathrm{div}\boldsymbol{\beta})\varphi v_h^2 - \frac{1}{2}\sum_{e\in\mathcal{E}_h}\int_e\{\boldsymbol{\beta}\varphi\}[\![v_h^2]\!]. \end{aligned} \tag{4.22}$$

Next, the continuity of $\boldsymbol{\beta}$ and φ easily imply that

$$\sum_{e\in\mathcal{E}_h^\circ}\int_e\{\boldsymbol{\beta}v_h\}\cdot[\![\varphi v_h]\!] = \frac{1}{2}\sum_{e\in\mathcal{E}_h^\circ}\int_e\{\boldsymbol{\beta}\varphi\}\cdot[\![v_h^2]\!].$$

From this and (2.10) we have then

$$\sum_{e\in\mathcal{E}_h^\circ}\int_e\{\boldsymbol{\beta}v_h\}_{upw}[\![\varphi v_h]\!] = \frac{1}{2}\sum_{e\in\mathcal{E}_h^\circ}\int_e\{\boldsymbol{\beta}\varphi\}\cdot[\![v_h^2]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e\frac{\boldsymbol{\beta}\cdot\mathbf{n}^+}{2}\varphi|[\![v_h]\!]|^2. \tag{4.23}$$

By noting that **(H1)** and (4.20) imply

$$-\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\varphi = (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\eta)\,\chi \ge 2b_0\chi \ge 2b_0\chi_1^*,$$

from (4.22)-(4.23), using (4.20), (2.2) and (4.4), we obtain:

$$\begin{aligned} a_h^{rc}(v_h, \varphi v_h) &= \int_\Omega[\gamma + \frac{1}{2}(\mathrm{div}\boldsymbol{\beta})]\varphi v_h^2 - \frac{1}{2}\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}\varphi)v_h^2 \\ &\quad + \sum_{e\in\mathcal{E}_h^\circ}\int_e\frac{\boldsymbol{\beta}\cdot\mathbf{n}^+}{2}\varphi|[\![v_h]\!]|^2 - \frac{1}{2}\int_{\Gamma^-}\boldsymbol{\beta}\cdot\mathbf{n}\varphi v_h^2 + \frac{1}{2}\int_{\Gamma^+}\boldsymbol{\beta}\cdot\mathbf{n}\varphi v_h^2 \\ &\ge \chi_1^*\|(\overline{\varrho} + b_0)^{1/2}v_h\|^2_{0,\Omega} + \frac{\alpha_1}{2}\sum_{e\in\mathcal{E}_h}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![v_h]\!]\|^2_{0,e} \ge \frac{\chi_1^*}{2}|||v_h|||^2_{rc}, \end{aligned}$$

that is (4.18). On the other hand (4.19) is again an easy consequence of (2.13) and (4.20)-(4.21). ∎

REMARK 4.1. *We point out that condition* (4.16) *has been taken in order to simplify the computation and to provide an easily quantifiable constant in* (4.17) *(very much in the spirit of* (4.12) *compared with the less demanding* (4.10)*). Looking at the proof, however, we see that we could stick to* (4.10) *(changing the conditions on* κ *in* (4.15) *in order to have* α_2/α_1 *as close to 1 as necessary). Hence, in some sense, the difficulty of finding "how big should* η_0 *be in practice" has not been worsened by the above trick.*

REMARK 4.2. *Concerning the other three methods* (3.6), (3.7), *and* (3.8), *they exhibit essentially the same terms, with the only exception for the method* (3.8), *where the advective part contains*

$$\sum_{e\in\mathcal{E}_h^\circ}\int_e ((\theta+1)\{\boldsymbol{\beta} v_h\}_\alpha - \theta\{\boldsymbol{\beta} v_h\})\,[\![\,\varphi v_h\,]\!] =: I_1,$$

instead of the left term in (4.23). *Using the definition* (2.9) *of the weighted average we obtain, instead of* (4.23)*:*

$$I_1 = \frac{1}{2}\sum_{e\in\mathcal{E}_h^\circ}\int_e \{\boldsymbol{\beta}\varphi\}\cdot[\![\,v_h^2\,]\!] + (\theta+1)\sum_{e\in\mathcal{E}_h^\circ}\int_e \frac{\boldsymbol{\beta}\cdot[\![\,\alpha\,]\!]}{2}\varphi|[\![\,v_h\,]\!]|^2$$

where $\boldsymbol{\beta}\cdot[\![\,\alpha\,]\!] = (2\alpha^+ - 1)\boldsymbol{\beta}\cdot\mathbf{n}^+ > 0$ *since* α^+, *the weight associated with the upwind triangle, is* $> 1/2$. *Hence,* (4.18) *holds also for method* (3.8) *(possibly with a different constant), if* $\theta > -1$. *As already said, choosing* $\theta = -1$ *in* (3.8) *produces undesirable cancellations which lead to have stability in a norm too weak to ensure control on the advective part. Namely, we have*

$$a_h(v_h, \varphi v_h) \geq C\,(\|(\overline{\varrho}+b_0)^{1/2} v_h\|_{0,\Omega}^2 + |||v_h|||_d^2 + \sum_{e\in\Gamma}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![\,v_h\,]\!]\|_{0,e}^2).$$

Suboptimal error estimates ($O(h^k)$) *in this norm can be obtained, but the method is unstable in strongly advective regimes. Indeed,* $\theta = -1$ *gives rise to a method without any kind of upwind.*

The following super-approximation results can be found in [23] and [31]. For convenience we briefly sketch the proof.

LEMMA 4.2. *Let* $\varphi \in W^{k+1,\infty}(\Omega)$ *be the function defined in* (4.14). *For* $v_h \in V_h^k$, *let* $\widetilde{\varphi v_h}$ *be the* L^2-*projection of* φv_h *in* V_h^k. *Then:*

$$\|\varphi v_h - \widetilde{\varphi v_h}\|_{0,\Omega} \leq C\frac{\|\chi\|_{k+1,\infty,\Omega}}{L} h\|v_h\|_{0,\Omega}, \tag{4.24}$$

$$|\varphi v_h - \widetilde{\varphi v_h}|_{1,h} \leq C\frac{\|\chi\|_{k+1,\infty,\Omega}}{L}\|v_h\|_{0,\Omega}, \tag{4.25}$$

$$(\sum_{e\in\mathcal{E}_h}\|\varphi v_h - \widetilde{\varphi v_h}\|_{0,e}^2)^{1/2} \leq C\frac{\|\chi\|_{k+1,\infty,\Omega}}{L} h^{1/2}\|v_h\|_{0,\Omega}, \tag{4.26}$$

where L *is the diameter of* Ω.

Proof. We shall deduce (4.24). Observe first that, since $\widetilde{\kappa v_h} \equiv \kappa v_h$,

$$\varphi v_h - \widetilde{\varphi v_h} \equiv \chi v_h - \widetilde{\chi v_h}.$$

Using classical interpolation results, the definition of the norm (1.2), the inverse inequality ([10, Theorem 17.2, pp.135]), and $h < L$ we have:

$$\begin{aligned}
\|\varphi v_h - \widetilde{\varphi v_h}\|_{0,T} &\leq C\, {h_T}^{k+1} |\chi v_h|_{k+1,T} \leq C {h_T}^{k+1} \sum_{j=0}^{k} |\chi|_{k+1-j,\infty,T} |v_h|_{j,T} \\
&\leq C \|\chi\|_{k+1,\infty,\Omega} \sum_{j=0}^{k} \frac{{h_T}^{k+1} |v_h|_{j,T}}{L^{k+1-j}} \\
&\leq C\, C_{inv} \frac{\|\chi\|_{k+1,\infty,\Omega}}{L} \|v_h\|_{0,T} \sum_{j=0}^{k} \frac{{h_T}^{k+1-j}}{L^{k-j}} \\
&\leq C\,(k+1) h_T \frac{\|\chi\|_{k+1,\infty,\Omega}}{L} \|v_h\|_{0,T}.
\end{aligned} \tag{4.27}$$

Hence, summing over all elements $T \in \mathcal{T}_h$ we reach (4.24). Exactly in the same way we prove (4.25), while (4.26) is a consequence of (4.24)-(4.25) via the trace inequality (2.12). □

LEMMA 4.3. *In the hypotheses of Lemma 4.1, there exist two positive constants χ_4^*, χ_5^* such that, for any value of κ, the corresponding φ verifies:*

$$a_h^d(v_h, \varphi v_h - \widetilde{\varphi v_h}) \leq \chi_4^* |||v_h|||_d^2 \quad \forall v_h \in V_h^k, \tag{4.28}$$

$$a_h^{rc}(v_h, \varphi v_h - \widetilde{\varphi v_h}) \leq \chi_5^* (\frac{h}{L})^{1/2} |||v_h|||_{rc}^2 \quad \forall v_h \in V_h^k. \tag{4.29}$$

Proof. Using estimates (4.25)-(4.26) from Lemma 4.2, and then (2.13) we see that

$$|||\widetilde{\varphi v_h} - \varphi v_h|||_d \leq C \frac{\|\chi\|_{k+1,\infty,\Omega}}{L} \varepsilon^{1/2} \|v_h\|_{0,\Omega} \leq C C_P \|\chi\|_{k+1,\infty,\Omega} |||v_h|||_d.$$

Hence, from (4.9) we have

$$a_h^d(v_h, \widetilde{\varphi v_h} - \varphi v_h) \leq C_d |||v_h|||_d |||\widetilde{\varphi v_h} - \varphi v_h|||_d \leq C_d C C_P \|\chi\|_{k+1,\infty,\Omega} |||v_h|||_d^2,$$

that is (4.28) with $\chi_4^* = C_d C C_P \|\chi\|_{k+1,\infty,\Omega}$. Before dealing with the reactive-convective part we observe that, if $P_h^0 \boldsymbol{\beta}$ is the L^2-projection of $\boldsymbol{\beta}$ onto constants, by definition of $\widetilde{\varphi v_h}$ it holds

$$\int_\Omega P_h^0 \boldsymbol{\beta} \cdot \nabla_h v_h (\varphi v_h - \widetilde{\varphi v_h}) = 0.$$

By integrating by parts and using (2.11) and (2.10) we have then:

$$\begin{aligned}
a_h^{rc}(v_h, \widetilde{\varphi v_h} - \varphi v_h) &= \int_\Omega [\gamma + \mathrm{div}\boldsymbol{\beta}] v_h (\widetilde{\varphi v_h} - \varphi v_h) + \int_\Omega [\boldsymbol{\beta} - P_h^0 \boldsymbol{\beta}] \cdot \nabla_h v_h (\widetilde{\varphi v_h} - \varphi v_h) \\
&- \sum_{e \notin \Gamma^+} \int_e \boldsymbol{\beta} \cdot [\![v_h]\!] \{\widetilde{\varphi v_h} - \varphi v_h\} + \sum_{e \in {\mathcal{E}_h}^o} \int_e \frac{\boldsymbol{\beta} \cdot \mathbf{n}^+}{2} [\![v_h]\!] [\![\widetilde{\varphi v_h} - \varphi v_h]\!] \\
&= I + II + III + IV.
\end{aligned}$$

From (2.2), (**H3**), and (2.4) we have:

$$
\begin{aligned}
I &= \int_\Omega \varrho v_h(\widetilde{\varphi v_h} - \varphi v_h) + \frac{1}{2}\int_\Omega \mathrm{div}\boldsymbol{\beta}\, v_h(\widetilde{\varphi v_h} - \varphi v_h)\\
&\le c_\varrho \|(\overline{\varrho}+b_0)^{1/2} v_h\|_{0,\Omega} \|(\overline{\varrho}+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|_{0,\Omega} + \frac{b_0}{2c_\beta}\|v_h\|_{0,\Omega}\|\widetilde{\varphi v_h} - \varphi v_h\|_{0,\Omega}.
\end{aligned}
$$

On the other hand, the definition (4.4) of $\overline{\varrho}$, and estimate (4.24) from Lemma 4.2 give

$$
\begin{aligned}
&\|(\overline{\varrho}+b_0)^{1/2}(\widetilde{\varphi v_h} - \varphi v_h)\|^2_{0,\Omega} = \sum_{T\in\mathcal{T}_h} (\overline{\varrho}_T + b_0)\|(\widetilde{\varphi v_h} - \varphi v_h)\|^2_{0,T}\\
&\le C\|\chi\|^2_{k+1,\infty,\Omega}\,(\frac{h}{L})^2 \sum_{T\in\mathcal{T}_h} (\overline{\varrho}_T + b_0)\|v_h\|^2_{0,T} = C\,\|\chi\|^2_{k+1,\infty,\Omega}(\frac{h}{L})^2\|(\overline{\varrho}+b_0)^{1/2}v_h\|^2_{0,\Omega},
\end{aligned}
$$

so that

$$
I \le C\|\chi\|_{k+1,\infty,\Omega}\frac{h}{L}\|(\overline{\varrho}+b_0)^{1/2}v_h\|^2_{0,\Omega}. \tag{4.30}
$$

Classical approximation results, (4.24), (2.4) and the inverse inequality give

$$
II \le Ch|\boldsymbol{\beta}|_{1,\infty,\Omega}|v_h|_{1,h}\frac{\|\chi\|_{k+1,\infty,\Omega}h}{L}\|v_h\|_{0,\Omega} \le C\|\chi\|_{k+1,\infty,\Omega}\frac{h}{L}\frac{b_0}{c_\beta}\|v_h\|^2_{0,\Omega}. \tag{4.31}
$$

Finally, from (4.26) we deduce

$$
\begin{aligned}
III + IV &\le C\,\frac{h^{1/2}}{L}\|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega}\|v_h\|_{0,\Omega}\Big(\sum_{e\in\mathcal{E}_h}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![\, v_h]\!]\|^2_{0,e}\Big)^{1/2}\|\chi\|_{k+1,\infty,\Omega}\\
&\le C\,(\frac{h}{L})^{1/2}(b_0\|v_h\|^2_{0,\Omega} + \sum_{e\in\mathcal{E}_h}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![\, v_h]\!]\|^2_{0,e})\|\chi\|_{k+1,\infty,\Omega}. \qquad (4.32)
\end{aligned}
$$

Collecting (4.30)-(4.31)-(4.32) we get then

$$
a_h^{rc}(v_h, \widetilde{\varphi v_h} - \varphi v_h) \le C\,\|\chi\|_{k+1,\infty,\Omega}(\frac{h}{L})^{1/2}|||v_h|||^2_{rc},
$$

that is (4.29) with $\chi_5^* = C\|\chi\|_{k+1,\infty,\Omega}$. ☐

The next theorem provides the first stability result for the variational formulations presented in Section 3.

THEOREM 4.4. *In the hypotheses of Lemma 4.1, there exists a positive constant* $\alpha_S = \alpha_S(\boldsymbol{\beta},\Omega)$*, and* $h_0 = h_0(\boldsymbol{\beta}) > 0$ *such that, for* $h < h_0$*:*

$$
\sup_{v_h\in V_h^k}\frac{a_h(u_h,v_h)}{|||v_h|||} \ge \alpha_S|||u_h||| \qquad \forall u_h \in V_h^k.
$$

Proof. For $u_h \in V_h^k$, let $v_h = \widetilde{\varphi u_h} \in V_h^k$ be the L^2-projection of φu_h as defined previously. We shall prove that

$$
|||v_h||| \le c_1|||u_h|||, \tag{4.33}
$$

$$
a_h(u_h, v_h) \ge c_2|||u_h|||^2. \tag{4.34}
$$

Adding and subtracting φu_h, from (4.17) we have first:

$$a_h^d(u_h, \widetilde{\varphi u_h}) = a_h^d(u_h, \widetilde{\varphi u_h} - \varphi u_h) + a_h^d(u_h, \varphi u_h)$$
$$\geq a_h^d(u_h, \widetilde{\varphi u_h} - \varphi u_h) + \frac{\chi_1^* + \kappa}{6} |||u_h|||_d^2.$$

Using estimate (4.28) we have then easily that for $\chi_1^* + \kappa$ bigger than $12\,\chi_4^*$ we find

$$a_h^d(u_h, \widetilde{\varphi u_h}) \geq \chi_4^* |||u_h|||_d^2.$$

In a similar way, from (4.29) and (4.18) one has, for $h < h_0$

$$a_h^{rc}(u_h, \widetilde{\varphi u_h}) \geq C\, |||u_h|||_{rc}^2,$$

with C depending only on χ_1^*, χ_5^*. On the other hand, using (4.19) and Lemma 4.2, we have easily

$$|||\widetilde{\varphi u_h}||| \leq c_1 |||u_h|||,$$

that is (4.33), with c_1 depending on χ_1^* and $\|\chi\|_{k+1,\Omega}$. □

Stability in a stronger norm

In a strongly advection dominated regime it is desirable to have a control also on the streamline derivative, that is, it is necessary to have in (4.3) a term of SUPG-type. We set:

$$|||v|||_{DG}^2 := |||v|||^2 + |||v|||_S^2, \quad |||v|||_S^2 = \sum_{T\in\mathcal{T}_h} \frac{h_T}{\|\boldsymbol{\beta}\|_{0,\infty,T}} \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} v)\|_{0,T}^2, \tag{4.35}$$

where P_h^k is again the L^2-projection on V_h^k.

Remark 4.3. *The presence of the projection in* (4.35) *is due to the fact that we assumed $\boldsymbol{\beta}$ to be a variable function, and hence $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h \notin V_h^k$. Clearly, whenever $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h \in V_h^k$, that is, if $\boldsymbol{\beta}$ is either constant ([19],[15],[9]) or piecewise linear ([17]), the projection can be removed.*

Stability in the norm (4.35) can be achieved again through an inf-sup condition.

Lemma 4.5. *There exists a constant $C_S > 0$, independent of $h, \varepsilon, \boldsymbol{\beta}, \gamma$ such that:*

$$\sup_{v_h\in V_h^k} \frac{a_h(u_h, v_h)}{|||v_h|||} \geq C_S(\|u_h\|_S - |||u_h|||) \qquad \forall u_h \in V_h^k. \tag{4.36}$$

Proof. For $u_h \in V_h^k$, let $P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h) \in V_h^k$ be the L^2-projection on V_h^k of $\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h$, for which the following estimates hold:

$$\forall T\in\mathcal{T}_h : |P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h)|_{1,T} \leq C_{inv} h_T{}^{-1} \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h)\|_{0,T}, \tag{4.37}$$

and, for any edge e, shared by two elements T^+ and T^-,

$$\begin{aligned} \|[\![P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)]\!]\|_{0,e}^2 &\leq C|e|^{-1} \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h)\|_{0,T^+\cup T^-}^2, \\ \|\{P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)\}\|_{0,e}^2 &\leq C|e|^{-1} \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h)\|_{0,T^+\cup T^-}^2. \end{aligned} \tag{4.38}$$

Inequality (4.37) is the usual inverse inequality, while (4.38) is deduced through the trace inequality (2.12), and (4.37). We then set $v_h = \sum_{T\in\mathcal{T}_h} c_T (P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h))_{|T}$, where

$$c_T = \begin{cases} \frac{h_T}{\|\boldsymbol{\beta}\|_{0,\infty,T}} & \text{if advection dominates in } T, \\ 0 & \text{otherwise.} \end{cases}$$

We shall prove that

$$|||v_h||| \leq C_1 \|u_h\|_S, \tag{4.39}$$

$$a_h(u_h, v_h) \geq C_2(\|u_h\|_S^2 - |||u_h||| \|u_h\|_S). \tag{4.40}$$

We prove first (4.39), having in mind that, if advection dominates, then

$$\varepsilon < h_T \|\boldsymbol{\beta}\|_{0,\infty,T}/2, \qquad \|\gamma + \mathrm{div}\boldsymbol{\beta}\|_{0,\infty,T} < \|\boldsymbol{\beta}\|_{0,\infty,T}/h_T \quad \forall T \in \mathcal{T}_h. \tag{4.41}$$

From (4.37) and (4.41) we deduce

$$\varepsilon |v_h|_{1,h}^2 = \sum_{T\in\mathcal{T}_h} \varepsilon \left(\frac{h_T}{\|\boldsymbol{\beta}\|_{0,\infty,T}} \right)^2 |P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)|_{1,T}^2 \leq C\, \|u_h\|_S^2. \tag{4.42}$$

Similarly, from (4.38) and (4.41) we have

$$\sum_{e\notin\Gamma_N} S_e \|[\![v_h]\!]\|_{0,e}^2 = \sum_{e\notin\Gamma_N} c_e \frac{\varepsilon}{|e|} \|[\![c_T P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)]\!]\|_{0,e}^2 \leq C\, \|u_h\|_S^2, \tag{4.43}$$

and

$$\sum_{e\in\mathcal{E}_h} \||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2} [\![v_h]\!]\|_{0,e}^2 = \sum_{e\in\mathcal{E}_h} \||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2} [\![c_T P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)]\!]\|_{0,e}^2 \leq C\, \|u_h\|_S^2. \tag{4.44}$$

Since $\varrho = (\gamma + \mathrm{div}\boldsymbol{\beta}) - \frac{1}{2}\mathrm{div}\boldsymbol{\beta}$, in view of (4.41) and (2.4) we deduce

$$\|\varrho\|_{0,\infty,T} \leq \|\gamma + \mathrm{div}\boldsymbol{\beta}\|_{0,\infty,T} + \frac{1}{2}\|\mathrm{div}\boldsymbol{\beta}\|_{0,\infty,T} \leq \frac{\|\boldsymbol{\beta}\|_{0,\infty,T}}{h_T} + \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{2L}.$$

Hence, from **(H2)** and since $h_T \leq h < L$ we deduce

$$c_T \|\varrho\|_{0,\infty,T} \leq 1 + \frac{h_T}{2Lc_\beta} \leq 1 + \frac{1}{2c_\beta}.$$

Consequently,

$$\|\overline{\varrho}^{1/2} v_h\|_{0,\Omega}^2 \leq \sum_{T\in\mathcal{T}_h} \|\varrho\|_{0,\infty,T} c_T^2 \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h)\|_{0,T}^2 \leq C\|u_h\|_S^2. \tag{4.45}$$

Finally, always from **(H2)**,

$$\|v_h\|_{0,\Omega}^2 = \sum_{T\in\mathcal{T}_h} \left(\frac{h_T}{\|\boldsymbol{\beta}\|_{0,\infty,T}} \right)^2 \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)\|_{0,T}^2 \leq \frac{h}{c_\beta \|\boldsymbol{\beta}\|_{1,\infty,\Omega}} \|u_h\|_S^2, \tag{4.46}$$

and then, since $b_0 = \|\boldsymbol{\beta}\|_{0,\infty,\Omega}/L$, $\|\boldsymbol{\beta}\|_{0,\infty,\Omega} \leq \|\boldsymbol{\beta}\|_{1,\infty,\Omega}$, and $h < L$,

$$b_0 \|v_h\|_{0,\Omega}^2 \leq \frac{1}{c_\beta} \|u_h\|_S^2.$$

This and (4.45) can be written as

$$\|(\overline{\varrho} + b_0)^{1/2} v_h\|_{0,\Omega}^2 \leq C\|u_h\|_S^2, \tag{4.47}$$

and (4.39) is proved. We turn now to prove (4.40), referring again to formulation (3.4). For the diffusive part we have, via Cauchy-Schwarz inequality and (4.42):

$$\int_\Omega \varepsilon \boldsymbol{\nabla}_h u_h \boldsymbol{\nabla}_h v_h \le \varepsilon^{1/2}|u_h|_{1,h}\varepsilon^{1/2}|v_h|_{1,h} \le C\varepsilon^{1/2}|u_h|_{1,h}\|u_h\|_S.$$

For the integrals on the edges, Cauchy-Schwarz inequality and (4.43) give:

$$\sum_{e\notin\Gamma_N} S_e \int_e [\![u_h]\!][\![v_h]\!] \le C\Big(\sum_{e\notin\Gamma_N} S_e\|u_h\|^2_{0,e}\Big)^{1/2}\|u_h\|_S \le C\|u_h\|_j\|u_h\|_S.$$

In an analogous way, Cauchy-Schwarz inequality, trace inequality (2.12), inverse inequality, and (4.43) give:

$$\sum_{e\notin\Gamma_N}\int_e\{\varepsilon\boldsymbol{\nabla}_h u_h\}\cdot[\![v_h]\!] \le C\,\varepsilon^{1/2}|u_h|_{1,h}\|u_h\|_S,$$

so that

$$a_h^d(u_h,w_h) \le C|\!|\!|u_h|\!|\!|\|u_h\|_S. \tag{4.48}$$

For the reactive and advective terms, integration by parts, formula (2.11) and the definition of the upwind average (2.10) give

$$\begin{aligned} a_h^{rc}(u_h,v_h) = &\int_\Omega \varrho u_h v_h + \int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)v_h + \frac{1}{2}\int_\Omega \mathrm{div}\boldsymbol{\beta} u_h v_h \\ &+\sum_{e\in\mathcal{E}_h^\circ}\int_e\frac{\boldsymbol{\beta}\cdot\mathbf{n}^+}{2}[\![u_h]\!][\![v_h]\!] - \sum_{e\notin\Gamma^+}\int_e\boldsymbol{\beta}\cdot[\![u_h]\!]\{v_h\}.\end{aligned}$$

By definition of projection we have:

$$\int_\Omega(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h)v_h = \int_\Omega P_h^k(\boldsymbol{\beta}\cdot\nabla_h u_h)\,v_h = \|u_h\|_S^2, \tag{4.49}$$

and by Cauchy-Schwarz inequality, **(H2)**, and (4.47):

$$\int_\Omega \varrho u_h v_h \le c_\varrho\|(\overline{\varrho}+b_0)^{1/2}u_h\|_{0,\Omega}\|(\overline{\varrho}+b_0)^{1/2}v_h\|_{0,\Omega} \le C\|(\overline{\varrho}+b_0)^{1/2}u_h\|_{0,\Omega}\|u_h\|_S. \tag{4.50}$$

Using (2.4), (4.46), and **(H2)** we obtain:

$$\begin{aligned}\int_\Omega \mathrm{div}\boldsymbol{\beta} u_h v_h &\le (\frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L})\|u_h\|_{0,\Omega}(\frac{h}{c_\beta\|\boldsymbol{\beta}\|_{1,\infty,\Omega}})^{1/2}\|u_h\|_S \\ &\le \frac{b_0^{1/2}}{c_\beta}(\frac{h}{L})^{1/2}\|u_h\|_{0,\Omega}\|u_h\|_S \le C|\!|\!|u_h|\!|\!|\|u_h\|_S.\end{aligned} \tag{4.51}$$

Finally, from Cauchy-Schwarz inequality and (4.44) we easily obtain:

$$\sum_{e\in\mathcal{E}_h}\int_e\frac{\boldsymbol{\beta}\cdot\mathbf{n}^+}{2}[\![u_h]\!][\![v_h]\!] \le C(\sum_{e\in\mathcal{E}_h^\circ}\||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2}[\![u_h]\!]\|^2_{0,e})^{1/2}\|u_h\|_S. \tag{4.52}$$

Collecting (4.49), (4.50), (4.51), and (4.52) we obtain:

$$a_h^{rc}(u_h, v_h) \geq \|u_h\|_S^2 - C|||u_h||| \|u_h\|_S.$$

From (4.48) and the above estimate we have then:

$$a_h(u_h, v_h) \geq \|u_h\|_S^2 - C|||u_h||| \|u_h\|_S,$$

which, together with (4.39) gives (4.36). □

THEOREM 4.6. *There exists a constant* $C_S = C_S(\boldsymbol{\beta}, \Omega) > 0$ *and* $h_0 = h_0(\boldsymbol{\beta}) > 0$, *such that for* $h < h_0$:

$$\sup_{v_h \in V_h^k} \frac{a_h(u_h, v_h)}{|||v_h|||} \geq C_S |||u_h|||_{DG} \qquad \forall u_h \in V_h^k.$$

Proof. The result follows from Theorem 4.4 and Lemma 4.5. □

We finally conclude by proving a result which provides stability in a norm of SUPG-type, but without the projection. However, this requires stronger regularity assumptions on $\boldsymbol{\beta}$, dictated by the polynomial degree. More precisely, when using V_h^k, we can prove stability in the norm:

$$|||u_h|||_{SS}^2 := |||u_h|||^2 + \|u_h\|_{\boldsymbol{\beta}}^2, \quad \text{with } \|u_h\|_{\boldsymbol{\beta}}^2 = \sum_{T \in \mathcal{T}_h} \frac{h_T}{\|\boldsymbol{\beta}\|_{0,\infty,T}} \|\boldsymbol{\beta} \cdot \nabla u_h\|_{0,T}^2, \tag{4.53}$$

only if $\boldsymbol{\beta} \in W^{k,\infty}(\Omega)$. In other words, our initial assumption $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)$ guarantees stability in the norm (4.53) only for piecewise linear approximations.

THEOREM 4.7. *Let* $\boldsymbol{\beta} \in W^{k,\infty}(\Omega)$, *being* $k \geq 1$ *the polynomial degree of* V_h^k. *Assume that,*

$$\exists\ c_\beta > 0 \text{ such that } |\boldsymbol{\beta}(x)| \geq c_\beta \|\boldsymbol{\beta}\|_{k,\infty,\Omega} \qquad \forall x \in \Omega. \tag{\textbf{H2a}}$$

Then, there exists a constant $C_{ss} = C_{ss}(\boldsymbol{\beta}, \Omega) > 0$, *and* $h_0 = h_0(\boldsymbol{\beta}) > 0$ *such that, for* $h < h_0$,

$$\sup_{v_h \in V_h^k} \frac{a_h(u_h, v_h)}{|||v_h|||} \geq C_{ss} |||u_h|||_{SS} \quad \forall u_h \in V_h^k. \tag{4.54}$$

Proof. The proof is accomplished by proceeding similarly as for Theorem 4.6 and we omit the details. Indeed, the only step that needs to be modified is (4.49), as all the others hold with the norm $\|\cdot\|_S$ replaced by $\|\cdot\|_{\boldsymbol{\beta}}$, by simply using the stability of the L^2-projection. By adding and subtracting $\sum_{T \in \mathcal{T}_h} c_T (\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h)_{|T}$ we find

$$\begin{aligned}\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h u_h) v_h &= \|u_h\|_{\boldsymbol{\beta}}^2 + \int_\Omega c_T (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h u_h)[P_h^k(\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h u_h) - \boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h u_h] \\ &\geq \|u_h\|_{\boldsymbol{\beta}}^2 - \|u_h\|_{\boldsymbol{\beta}} (\sum_{T \in \mathcal{T}_h} c_T \|P_h^k(\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h) - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h\|_{0,T}^2)^{1/2}.\end{aligned}$$

To estimate the second term, note that the regularity of $\boldsymbol{\beta}$ allows to use the super-approximation property (4.27) (with $\boldsymbol{\beta}$ now playing the role of φ, and ∇u_h playing the role of v_h). This plus inverse inequality and (**H2a**) gives:

$$\begin{aligned}\|P_h^k(\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h) - \boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h\|_{0,T} &\leq C {h_T}^k |\boldsymbol{\beta} \cdot \boldsymbol{\nabla} u_h|_{k,T} \leq Ck \frac{\|\boldsymbol{\beta}\|_{k,\infty,\Omega}}{L} h_T \|\boldsymbol{\nabla} u_h\|_{0,T} \\ &\leq C \frac{\|\boldsymbol{\beta}\|_{k,\infty,\Omega}}{L} \|u_h\|_{0,T} \leq C \frac{\|\boldsymbol{\beta}\|_{0,\infty,T}}{c_\beta L} \|u_h\|_{0,T}.\end{aligned}$$

Since $h < L$ we have then

$$\sum_{T\in\mathcal{T}_h} c_T \|P_h^k(\boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h) - \boldsymbol{\beta}\cdot\boldsymbol{\nabla} u_h\|_{0,T}^2 \le C \sum_{T\in\mathcal{T}_h} (\frac{h_T}{L}) \frac{\|\boldsymbol{\beta}\|_{0,\infty,T}}{c_\beta^2 L} \|u_h\|_{0,T}^2 \le \frac{C}{c_\beta^2} b_0 \|u_h\|_{0,\Omega}^2.$$

Thus,

$$\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h u_h) v_h \ge \|u_h\|_{\boldsymbol{\beta}}^2 - C\,\|u_h\|_{\boldsymbol{\beta}} |||u_h|||.$$

Then, the result (4.54) follows. ◻

5. A priori error estimates. We next show a-priori error estimates in the norms (4.3) and (4.35) for the methods presented. Let P_h^k be the L^2-projection in V_h^k, for which the following local approximation property hold

$$\|u - P_h^k u\|_{r,T} \le C h^{k+1-r} |u|_{k+1,T}, \quad r = 0,1,2, \qquad T \in \mathcal{T}_h\ , \tag{5.1}$$

$$\|u - P_h^k u\|_{r,p,T} \le C h^{k+1-r} |u|_{k+1,p,T},\ \ 1 \le p \le \infty,\ \ r = 0,1, \quad T \in \mathcal{T}_h\ . \tag{5.2}$$

Moreover, from (5.1)-(2.12) we deduce that

$$\|u - P_h^k u\|_{0,e} \le C h_T^{k+1/2} |u|_{k+1,T} \qquad \forall e \in \mathcal{E}_h. \tag{5.3}$$

Theorem 5.1. *Let u be the solution of (2.1), and let u_h be the solution of the discrete problems (4.1). There exists a constant $C_0 = C_0(\Omega)$, depending on the domain Ω, the shape regularity of $\mathcal{T}_h$ and the polynomial degree (but independent of h and the coefficients of the problem), such that:*

$$|||u - u_h||| \le C_0(\Omega) h^k \left(\varepsilon^{1/2} + \|\boldsymbol{\beta}\|_{0,\infty,\Omega}^{1/2} h^{1/2} + \|\varrho\|_{0,\infty,\Omega}^{1/2} h\right). \tag{5.4}$$

Proof. We define

$$\eta = u - P_h^k u, \qquad \delta = u_h - P_h^k u.$$

From Theorem 4.4 and Galerkin orthogonality (4.2) we have

$$\alpha_S |||\delta||| \le \frac{a_h(\delta, v_h)}{|||v_h|||} = \frac{a_h(\eta, v_h)}{|||v_h|||}. \tag{5.5}$$

The diffusive part is standard, and can be easily estimated through the trace inequality (2.12), (5.1), and (5.3):

$$a_h^d(\eta, v_h) \le C h^k \varepsilon^{1/2} |u|_{k+1,\Omega} |||v_h|||_d. \tag{5.6}$$

Regarding the advective part, since $P_h^0\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v_h \in V_h^k$, by definition of projection

$$\int_\Omega P_h^0\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v_h \eta = 0.$$

From this, the Cauchy-Schwarz inequality, (5.2), the inverse inequality, (2.4), and (5.1) we have:

$$\begin{aligned}\int_\Omega -(\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v_h)\eta &= \int_\Omega (P_h^0\boldsymbol{\beta} - \boldsymbol{\beta})\cdot\boldsymbol{\nabla}_h v_h \eta \le C h |\boldsymbol{\beta}|_{1,\infty,\Omega} |v_h|_{1,h} \|\eta\|_{0,\Omega} \\ &\le C \frac{\|\boldsymbol{\beta}\|_{1,\infty,\Omega}}{L} \|v_h\|_{0,\Omega} \|\eta\|_{0,\Omega} \le C \frac{b_0}{c_\beta} \|v_h\|_{0,\Omega} h^{k+1} |u|_{k+1,\Omega} \\ &\le C h^{k+1} b_0^{1/2} |u|_{k+1,\Omega} |||v_h||| = C \left(\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L}\right)^{1/2} h^{k+1} |u|_{k+1,\Omega} |||v_h|||.\end{aligned} \tag{5.7}$$

Using (5.3) we obtain:

$$\sum_e \int_e \{\boldsymbol{\beta}\eta\} \cdot [\![v_h]\!] \leq \|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega} \sum_e \|\{\eta\}\|_{0,e} \||\boldsymbol{\beta}\cdot\mathbf{n}|^{1/2} [\![v_h]\!]\|_{0,e}$$
$$\leq C\|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega} h^{k+1/2}|u|_{k+1,\Omega}|\!|\!| v_h |\!|\!|, \tag{5.8}$$

and arguing similarly, we have

$$\sum_e \int_e \frac{\boldsymbol{\beta}\cdot\mathbf{n}^+}{2} [\![\eta]\!] \cdot [\![v_h]\!] \leq C\|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega} h^{k+1/2}|u|_{k+1,\Omega}|\!|\!| v_h |\!|\!|. \tag{5.9}$$

Finally, by writing $\gamma = \varrho - \mathrm{div}\boldsymbol{\beta}/2$, using **(H3)**, (2.4), and (5.1) we obtain:

$$\begin{aligned}\int_\Omega \gamma\eta v_h &\leq \|\varrho\|^{1/2}_{0,\infty,\Omega}\|\eta\|_{0,\Omega} c_\rho^{1/2}\|(\overline{\varrho}+b_0)^{1/2}v_h\|_{0,\Omega} + \frac{b_0^{1/2}}{c_\beta}\|\eta\|_{0,\Omega} b_0^{1/2}\|v_h\|_{0,\Omega} \\ &\leq C\, h^{k+1}(\|\varrho\|^{1/2}_{0,\infty,\Omega} + (\frac{\|\boldsymbol{\beta}\|_{0,\infty,\Omega}}{L})^{1/2})|u|_{k+1,\Omega}|\!|\!| v_h |\!|\!|.\end{aligned} \tag{5.10}$$

Collecting then (5.6)—(5.10) and using $h/L < 1$ we obtain

$$a_h(\eta, v_h) \leq Ch^k\left(\varepsilon^{1/2} + \|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega}\, h^{1/2} + \|\varrho\|^{1/2}_{0,\infty,\Omega}\, h\right)|u|_{k+1,\Omega}|\!|\!| v_h |\!|\!|.$$

Hence, substituting this estimate into (5.5) gives

$$|\!|\!| \delta |\!|\!| \leq C(\Omega)h^k\left(\varepsilon^{1/2} + \|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega}\, h^{1/2} + \|\varrho\|^{1/2}_{0,\infty,\Omega}\, h\right)|u|_{k+1,\Omega}.$$

The result (5.4) then follows by triangle inequality. □

THEOREM 5.2. *Let u be the solution of* (2.1)*, and let u_h be the solution of the discrete problems (4.1). There exists a constant $C_1 = C_1(\Omega)$, depending on Ω, the shape regularity of $\mathcal{T}_h$ and the polynomial degree (but independent of γ, $\boldsymbol{\beta}$, ε, and h), such that:*

$$|\!|\!| u - u_h |\!|\!|_{DG} \leq C_1(\Omega)h^k\left(\varepsilon^{1/2} + \|\boldsymbol{\beta}\|^{1/2}_{0,\infty,\Omega}\, h^{1/2} + \|\varrho\|^{1/2}_{0,\infty,\Omega}\, h\right)|u|_{k+1,\Omega}.$$

Proof. The proof follows the same steps of Theorem 5.1, using the stability result of Theorem 4.6. Hence we omit the details. □

REMARK 5.1. *The same error estimates hold in the norm $|\!|\!| \cdot |\!|\!|_{SS}$ under the assumption $\boldsymbol{\beta} \in W^{k,\infty}(\Omega)$.*

REMARK 5.2. *Theorems* (5.1) *and* (5.2) *provide robust a-priori error estimates, which are optimal in all regimes. More precisely, we have*

$$|\!|\!| u - u_h |\!|\!|\ ,\ |\!|\!| u - u_h |\!|\!|_{DG} \simeq \begin{cases} O(h^{k+1/2}) & \text{if advection dominates,} \\ O(h^k) & \text{if diffusion dominates,} \\ O(h^{k+1}) & \text{if reaction dominates.} \end{cases}$$

COROLLARY 5.3. *As a direct consequence of our error analysis we have the following result:*

$$\|u - u_h\|_{0,\Omega} \leq C_2|u|_{k+1,\Omega}\begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates,} \\ h^{k+1} & \text{if reaction dominates,} \end{cases} \tag{5.11}$$

where C_2 depends on the domain Ω, the shape regularity of $\mathcal{T}_h$, the polynomial degree and on the coefficients of the problem γ, $\boldsymbol{\beta}$, and ε (but is independent of h).

REMARK 5.3. *Estimate* (5.11) *is suboptimal in the diffusion dominated regime, since it was simply obtained through* (2.13) *and* (5.4)*. In the advection dominated regime, although suboptimal of* 1/2*, is the best that one can expect for a regular triangulation without any further assumption on the construction-orientation of the mesh (see [24] for a counterexample in the pure hyperbolic case). Improved estimates in the case of $\boldsymbol{\beta}$ constant have been rigourously shown in [25] (for the pure hyperbolic case) under certain restrictions on the mesh, and more recently in [13], under milder assumptions on the grid. The techniques used in these papers rely strongly on the hypothesis that $\boldsymbol{\beta}$ is constant, and do not seem to be easily extendable to the case of variable $\boldsymbol{\beta}$. However, as we shall see in the next section, in many test cases optimal order of convergence in L^2 is attained for quite general mesh partitions.*

6. Numerical Experiments. In this section we compare on various test problems the methods analyzed in the previous sections. All the experiments where performed on the unit square $\Omega = (0,1)^2$, using piecewise linear approximations on triangular grids, structured and unstructured. In all the graphics, method (3.4) is represented by $- \cdot - \star \cdot - \cdot -$; method (3.6) with $- - \square - -$; method (3.7) with $\cdots \circ \cdots$ and method (3.8) with $-$x$-$. All the computations were done in Matalb7, on a Powerbook 1.5 with 2Gb of Ram memory.

Example 1: Case of Smooth Solution
We take $\boldsymbol{\beta} = [1,1]^T$, $\gamma = 0$ and we vary the diffusion coefficient $\varepsilon = 1, 10^{-3}, 10^{-9}$. The forcing term f is chosen so that the analytical solution of (2.1), with Dirichlet boundary conditions, is given by $u(x,y) = \sin(2\pi x)\sin(2\pi y)$. Fig. 6.1 and Fig. 6.2 represent the convergence diagrams in the norm $|||\cdot|||_{DG}$ (and $|||\cdot|||$, resp.). Clearly,

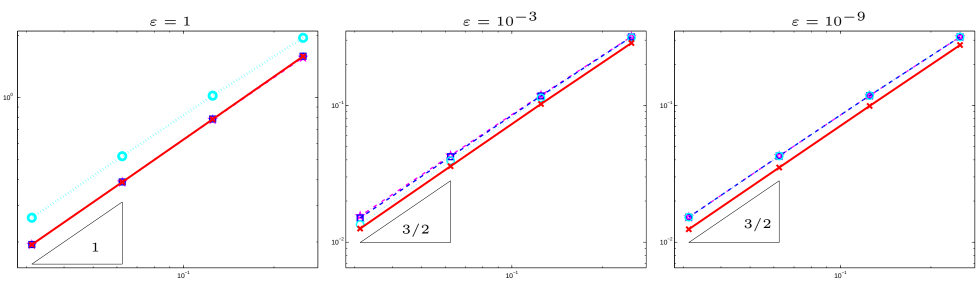

FIG. 6.1. *Example 1. Convergence Diagrams in the $|||\cdot|||_{DG}$-norm. Unstructured grids.*

the convergence rates are the same for all the methods, in agreement with the theory of Section 5: first order accuracy when diffusion dominates and order 3/2 in the convection dominated regime. In Fig. 6.3 are depicted the convergence diagrams in the L^2-norm, on structured grids. Similar results, although not reported here, were obtained on unstructured grids. Observe that, due to smoothness of the solution, second order of convergence is attained in all regimes for all the methods but method (3.7), which is only first order accurate when diffusion dominates. This is due to the fact that in the method (3.7) upwind is done on the whole flux. In method (3.8) the whole flux is also upwinded, but the use of the weighted average (2.9) allows to tune the amount of upwind as a function of the data.

Example 2: Rotating Flow

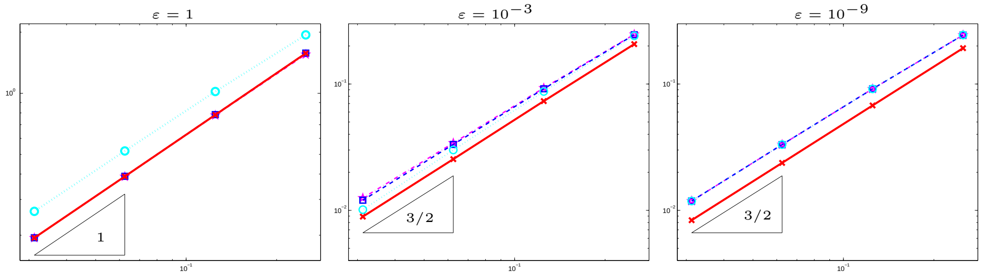

FIG. 6.2. *Example 1. Convergence Diagrams in the $|||\cdot|||$-norm. Unstructured grids.*

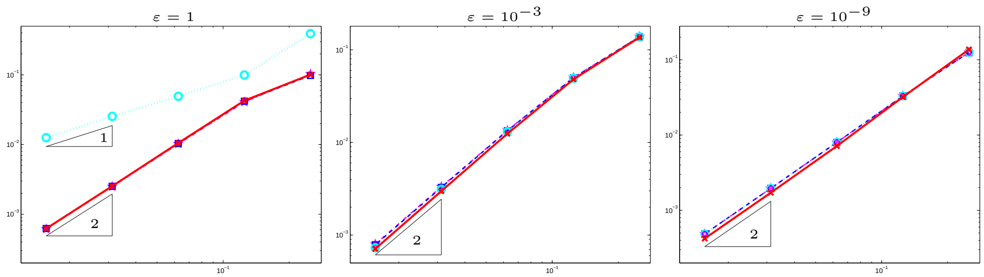

FIG. 6.3. *Example 1. Convergence Diagrams in the L^2-norm. Structured grids.*

This example is taken from [18]. The data are $\gamma = 0$, $\boldsymbol{\beta} = [y - 1/2, 1/2 - x]^T$, and no external forces act on the system. The solution u is prescribed along the slit $1/2 \times [0, 1/2]$, as follows:

$$u(1/2, y) = \sin^2(2\pi y) \qquad y \in [0, 1/2] \ .$$

In Fig. 6.4, for $\varepsilon = 10^{-9}$, we have represented the approximate solution obtained with the four methods on a structured triangular grid of 512 elements. As it can be seen, all the methods perform similarly, and no significant differences can be appreciated. An important feature of all the methods is the absence of crosswind diffusion which occurs with stabilized conforming methods. To better assess this feature of the methods, we have plotted in Fig. 6.5 the profile of the approximate solutions at $y = 1/2$.

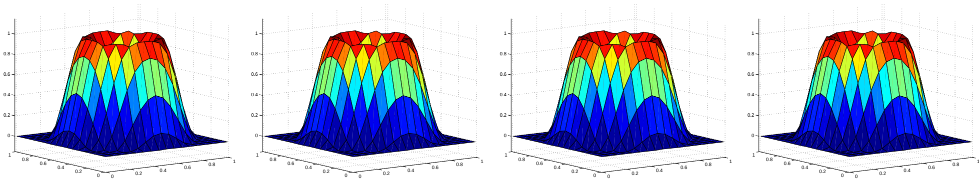

FIG. 6.4. *Example 2. Approximate solutions for $\varepsilon = 10^{-9}$ on structured grids. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

Example 3. Internal Layers

The next example is devoted to asses the performance of the methods in the presence of interior layers. We set $\gamma = 0$, $\boldsymbol{\beta} = [1/2, \sqrt{3}/2]^T$, and Dirichlet boundary conditions

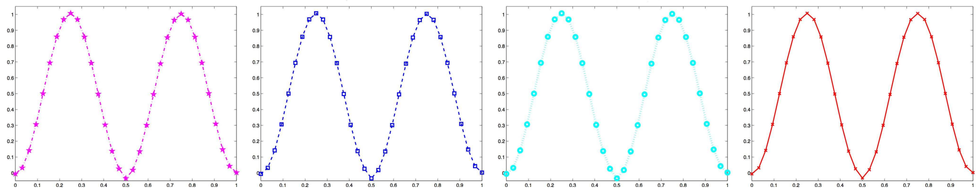

FIG. 6.5. *Example 2. Profile of the approximate solutions at* $y = 1/2$*;* $\varepsilon = 1\mathrm{e}-07$ *. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

as follows:

$$u = \begin{cases} 1 & \text{on } \{y = 0,\ 0 \le x \le 1\}, \\ 1 & \text{on } \{x = 0,\ y \le 1/5\}, \\ 0 & \text{elsewhere.} \end{cases}$$

The diffusion coefficient is varied from $\varepsilon = 10^{-3}$ to the limit case $\varepsilon = 0$ (pure-hyperbolic case). In Fig. 6.6 are represented the approximate solutions obtained on structured grids of 512 triangles with all methods for $\varepsilon = 10^{-3}$. They all behave poorly in the intermediate regimes, as they produce wiggles close to the boundary. These oscillations disappear in the advection-dominated regime (see Fig. 6.7), and the internal layer is sharply captured, with very small overshooting/undershooting. This can be better observed in Fig. 6.8, where we have represented the profiles of the solutions at $x = 0$. Similar result were observed for the profiles at $y = 0.5$.

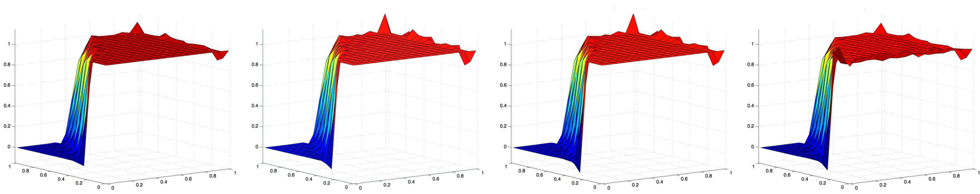

FIG. 6.6. *Example 3. Approximate solutions for* $\varepsilon = 10^{-3}$ *on unstructured grids. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

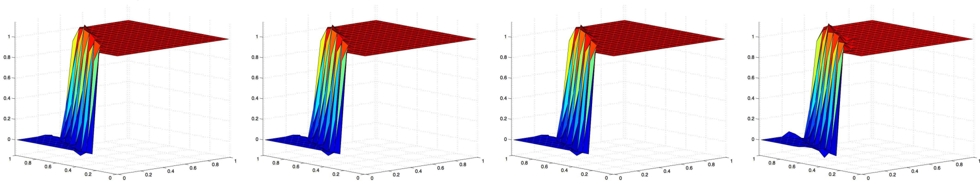

FIG. 6.7. *Example 3. Approximate solutions for* $\varepsilon = 10^{-9}$ *on unstructured grids. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

Example 4. Boundary Layers

In this last example we apply the methods to a boundary layer problem taken from [17]. The data are $\gamma = 0$, $\boldsymbol{\beta} = [1, 1]^T$, and we again vary the diffusion coefficient ε.

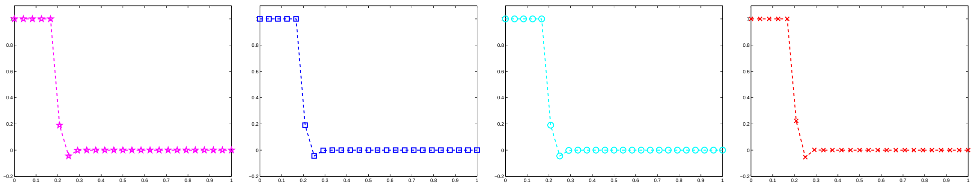

FIG. 6.8. *Example 3. Profile of the approximate solutions at $x = 0$; $\varepsilon = 1e-09$. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

The forcing term f is chosen so that the exact solution is given by:

$$u(x,y) = x + y(1-x) + \frac{e^{-1/\varepsilon} - e^{-(1-x)(1-y)/\varepsilon}}{1 - e^{-1/\varepsilon}}\ , \quad (x,y) \in \Omega\ .$$

This problem can be regarded as a multidimensional variant of the one-dimensional problem considered by Melenk et al. in [21]. Unlike the classical test case [32], u does not reduce, in the hyperbolic limit case, to a linear function in the interior of the domain, as shown in Fig. 6.9 (left), for $\varepsilon = 10^{-9}$. In the same figure (right)

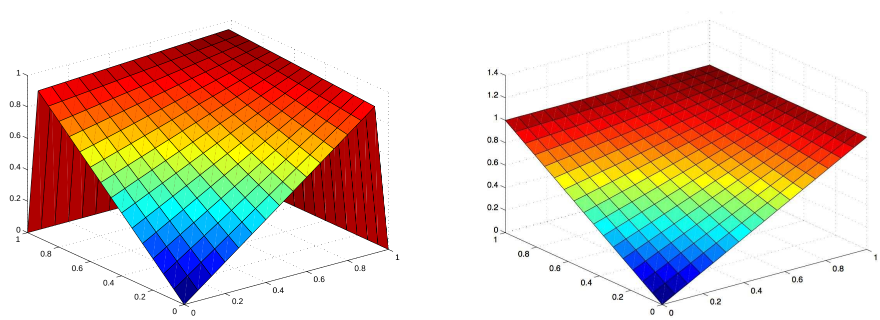

FIG. 6.9. *Example 4. Exact solution (left), approximate solution with method (3.7) (right); $\varepsilon = 10^{-9}$.*

only the solution obtained with the method (3.7) is represented, as all the methods do not exhibit visible differences in the strongly advective regime. Notice that, since boundary conditions are imposed in a weak way, the boundary layer is not captured by the DG-approximations, although the solution is free of spurious oscillations. In

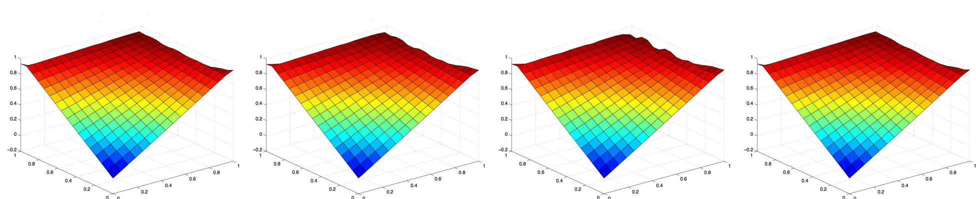

FIG. 6.10. *Example 4. Approximate solutions for $\varepsilon = 10^{-3}$. From left to right: methods (3.4), (3.6), (3.7), and (3.8).*

Fig. 6.10 we compare the methods for $\varepsilon = 10^{-3}$ and structured grids with 24x24x2 triangles. Again, no substantial differences can be observed, except for small oscillations in the method (3.7) (third plot in the figure), probably due to the upwind treatment of the diffusive part of the flux. For this test case we chose not to plot convergence diagrams in the norms (4.3) or (4.35) since, due to the weak approximation of the boundary conditions, the main contribution to the error comes from the error in the boundary layer, which is $O(1)$, as it can be seen in Figures 6.9 and 6.10. Fig. 6.11 represents the convergence diagrams in the L^1-norm for $\varepsilon = 10^{-3}$. Note that as we would expect in this regime, and since we are measuring global errors, first order of convergence is achieved. Although there are no great differences between the methods, it seems that in this case method (3.4) gives the most accurate approximation. This can also be checked from Figure 6.10. Finally, Figure 6.12 shows the convergence diagrams on unstructured grids, for $\varepsilon = 10^{-9}$ in the L^2-norm (left), the $|||\cdot|||_d$–norm in the interior of the domain, (i.e., without the contribution of the boundary elements) (center), and in the norm $|||\cdot|||_S$ defined in (4.35) (right). Note that all the methods give optimal order of convergence in L^2 in the advection dominated regime(see Remark 5.3).

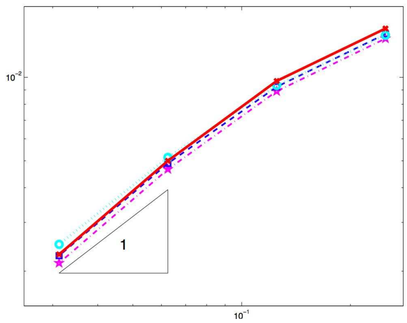

FIG. 6.11. *Example 4. Convergence diagrams in the L^1-norm, $\varepsilon = 10^{-3}$.*

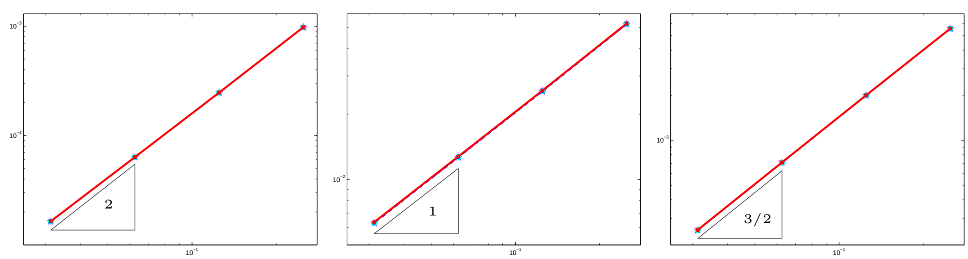

FIG. 6.12. *Example 4. Convergence diagrams in the norms L^2 (left), interior $|||\cdot|||_d$ (center), and $|||\cdot|||_S$ (right); $\varepsilon = 10^{-9}$. Unstructured grids.*

7. Conclusions. By using the weighted-residual approach of [7] we set a unified framework for deriving and analyzing various methods for advection-diffusion-reaction problems. The analysis carried out applies to the case of variable convection and reaction fields, and shows that optimal estimates in $DG-$norms are achieved. All the methods considered in this paper seem to have the same stability and accuracy properties, in all regimes. This is also confirmed numerically, though the method (3.8) seems to be more flexible in the intermediate regimes, thanks to the possibility of tuning the amount of upwind.

Appendix A. We briefly sketch how the function $\eta \in W^{k+1,\infty}(\Omega)$ in **(H1)** can be constructed. Arguing as in [14] we can guarantee that for $\boldsymbol{\beta}$ satisfying (2.3),

$$\text{if } \boldsymbol{\beta} \in [W^{1,\infty}(\Omega)]^2 \implies \exists\ \widetilde{\eta} \in W^{1,\infty}(\Omega) \quad \text{s.t.} \quad \boldsymbol{\beta}\cdot\nabla\widetilde{\eta} \geq 2b_0 > 0 \quad \text{in} \quad \Omega. \qquad (A.1)$$

We next show how from this function η_0, the more regular η in **(H1)** can be constructed. Let $\{\mathcal{U}_\alpha^+\}_\alpha$ be a finite open covering of Ω such that each $\mathcal{U}_\alpha^+$ enjoys the following property: there exists some $\varepsilon_1 > 0$ (to be chosen later) such that

$$\text{if} \quad x, y \in \mathcal{U}_\alpha^+ \quad \Longrightarrow \quad \|\boldsymbol{\beta}(x) - \boldsymbol{\beta}(y)\|_{0,\infty} < \varepsilon_1, \tag{A.2}$$

$$\text{and} \quad \forall\, x, y \in \mathcal{U}_\alpha^+ \qquad \boldsymbol{\beta}(x) \cdot \nabla\widetilde{\eta}(y) \geq b_0. \tag{A.3}$$

Inequality (A.3) is actually a consequence of (A.2) and (A.1). Indeed:

$$\boldsymbol{\beta}(x) \cdot \nabla\widetilde{\eta}(y) = \boldsymbol{\beta}(y) \cdot \nabla\widetilde{\eta}(y) + [\boldsymbol{\beta}(x) - \boldsymbol{\beta}(y)] \cdot \nabla\widetilde{\eta}(y) \geq 2b_0 - \varepsilon_1 \|\nabla\widetilde{\eta}\|_{0,\infty}.$$

Hence, by taking $\varepsilon_1 = b_0/\|\nabla\widetilde{\eta}\|_{0,\infty}$ one can guarantee (A.3). Let $\mathcal{U}_{\alpha'}^- \subset \mathcal{U}_\alpha^+$ be such that (A.2) holds with such choice of ε_1 (so that (A.3) is valid for all x and $y \in \mathcal{U}_\alpha^-$), and such that $\{\mathcal{U}_{\alpha'}^-\}_{\alpha'}$ is still an open covering of Ω. Next, on each $\mathcal{U}_{\alpha'}^-$ we mollify $\widetilde{\eta}$ by convolution with some ρ_δ mollifier; $\eta_{\alpha'}^\delta = \widetilde{\eta} * \rho_\delta$ in $\mathcal{U}_{\alpha'}^-$. Then, by taking a partition of unity $\{\phi_{\alpha'}\}_{\alpha'}$ associated to the covering $\{\mathcal{U}_{\alpha'}^-\}_{\alpha'}$ we can construct η as in **(H1)** by gluing the mollified $\eta_{\alpha'}^\delta$, that is $\eta = \sum_{\alpha'} \eta_{\alpha'}^\delta \cdot \phi_{\alpha'}$. Thus, the existence of η sufficiently smooth satisfying **(H1)** is guaranteed .

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