Stabilization of Galerkin methods and applications to domain decomposition

C. Baiocchi[⋆†], F. Brezzi[⋆†], L.D. Marini[‡†]

[⋆] Università degli Studi, 27100 PAVIA (Italy)

[†] Istituto di Analisi Numerica del C.N.R., 27100 Pavia (Italy)

[‡] Università di Genova, 16132 Genova (Italy)

Abstract. We present an abstract stabilization method which covers previous concrete applications to advection–diffusion equations and to the Stokes equations for incompressible fluids. We then apply the method to stabilize domain decomposition formulations for elliptic problems. We obtain a method that allows the treatment of internal variables, interface variables and Lagrange multipliers (normal derivatives) by piecewise polynomials of arbitrary order.

Introduction

We present an abstract regularization result inspired essentially by previous regularization techniques introduced by Hughes and various other authors for advection-diffusion problems and for the Stokes equations for incompressible fluids (see [12], [11], [10] for surveys and references). To be more precise, the application of our abstract result to the Stokes equations leads to the method introduced by Douglas and Wang [9], while the application to advection-diffusion problems produces a variant of Galerkin least square methods studied in [11]. For a more general abstract setting which includes other regularization techniques, see [2]. If we apply our result to the Dirichlet problem (for linear elliptic equations) with Lagrange multipliers we obtain a variant of [3].

In this paper the use of this abstract theory for the macro-hybrid domain decomposition method of [6],[7] is investigated. We recall that the method of [6] produces a three-field formulation where the three unknowns represent the solution u of the original problem inside each macro-element, the normal derivative λ of u on the boundary of each macro-element, and the trace ψ of u at the interfaces. The application of our abstract theory to this method allows a great generality in the choice of discretizations for the three fields. For instance, piecewise polynomials of arbitrary (and independent) degree can be chosen for each variable, still preserving stability and optimal error bounds. Alternatively, different Galerkin methods (finite elements, spectral, Fourier, wavelets etc.) can be used in different macro-elements to obtain a variant of the mortar elements techniques of [4].

An outline of the paper is as follows. In Section 1, starting from an abstract variational formulation, we present a class of regularization techniques and prove stability and error bounds under reasonable assumptions. To help the reader, the

Douglas-Wang method for Stokes is used throughout this section to clarify the abstract objects we have to deal with. In Section 2 we recall the macro-hybrid formulation of [6] for domain decomposition methods. In Section 3 we briefly apply the results of Section 1 to the three-field formulation of Section 2 and we sketch the error bounds that can be obtained. We refer to [8] for a more detailed treatment.

1 An Abstract Result

Let $\mathcal{V}_1$ be a Hilbert space, and $\mathcal{A}_1(U, V)$ a bilinear form on $\mathcal{V}_1 \times \mathcal{V}_1$, which is continuous in the usual sense:

$$\exists\, M_1 > 0 \qquad \mathcal{A}_1(U, V) \leq M_1 \, ||U||_1 \, ||V||_1 \,. \tag{1.1}$$

We assume that there exist two positive constants β_r and β_l such that for every W in $\mathcal{V}_1$ there exist V_r and V_l in $\mathcal{V}_1 - \{0\}$ with

$$\mathcal{A}_1(V_l, W) \geq \beta_l \, ||V_l||_1 \; ||W||_1 \,, \tag{1.2}$$

$$\mathcal{A}_1(W, V_r) \geq \beta_r \, ||W||_1 \; ||V_r||_1 \,. \tag{1.3}$$

It is well known that, with these assumptions, for every $f_1 \in \mathcal{V}_1'$ there exists a unique U_1 in $\mathcal{V}_1$ such that

$$\mathcal{A}_1(U_1, V) = < f_1, V > \quad \forall V \in \mathcal{V}_1 \,. \tag{1.4}$$

Remark 1.1: A tipical example of the situation that we are going to face is given by the Stokes problem, which can be fit in our framework by setting, with obvious notation:

$$\begin{cases} \mathcal{V}_1 = \left(H_0^1(\Omega)\right)^2 \times \left(L^2(\Omega)/\mathbb{R}\right) ; U = (\underline{u}, p); V = (\underline{v}, q) \\ \mathcal{A}_1(U, V) = \left(\underline{\underline{\nabla}}\, \underline{u}, \underline{\underline{\nabla}}\, \underline{v}\right) - (\text{ div}\, \underline{v}, p) + (\text{ div}\, \underline{u}, q). \end{cases} \tag{1.5}$$

The well–posedness of (1.4) follows from the inf–sup condition (see e.g. [5]) and cannot be deduced using Lax–Milgram theorem. ∎

Let now $\{\mathcal{V}_h\}_{h>0}$ be a sequence of finite dimensional subspaces of $\mathcal{V}_1$. It is well known that, in general, the problem of finding U_h^1 in $\mathcal{V}_h$ such that

$$\mathcal{A}_1\left(U_h^1, V_h\right) = < f_1, V_h > \qquad \forall V_h \in \mathcal{V}_h \tag{1.6}$$

does not have a unique solution, unless discrete analogues of (1.2) and (1.3) hold. We present here an abstract regularization technique that allows to circumvent this problem. When applied to the Stokes problem our technique reproduces the Douglas–Wang method (see [9]); when applied to the domain decomposition formulation of [6] it gives rise, as we shall see, to a variant of [3]. Although this last application is our aim here, we shall keep track of the Stokes problem as a first illustration of the abstract theory. The main reason for that is that Stokes problem is much more familiar and also formally simpler.

First of all we assume that the bilinear form $\mathcal{A}_1(U,V)$ is non–negative. More precisely, we assume that there exist a seminorm $|\cdot|$ on $\mathcal{V}_1$ and a positive constant γ_1 such that

$$\mathcal{A}_1(V,V) \geq \gamma_1 |V|^2 \quad \forall V \in \mathcal{V}_1 . \tag{1.7}$$

We also assume that

$$|V| \leq ||V||_1 \qquad \forall V \in \mathcal{V}_1 . \tag{1.8}$$

For the Stokes problem we clearly have $|V| = |(\underline{v},q)| = ||\underline{\underline{\nabla}}\,\underline{v}||^2_{L^2(\Omega)}$.

We assume now that we are given a sequence $\{\mathcal{V}_2(h)\}_{h>0}$ of Hilbert spaces, with seminorms $|V|_{2,h}$ and norms $||V||_{2,h}$, such that $\mathcal{V}_h \subset \mathcal{V}_2(h) \hookrightarrow V_1$ and U_1 (solution of (1.4)) belongs to $\mathcal{V}_2(h)$ for every h; we also assume that

$$||V_h||_{2,h} \leq M_e ||V_h||_1 \qquad \forall\, V_h \in \mathcal{V}_h \tag{1.9}$$

$$||V_h||^2_{2,h} \leq c\left(|V_h|^2 + |V_h|^2_{2,h}\right) \qquad \forall\, V_h \in \mathcal{V}_h \tag{1.10}$$

with M_e, c constants independent of h.

For the Stokes problem and for piecewise polynomial spaces $\mathcal{V}_h$, we can take

$$||(\underline{v},q)||^2_{2,h} = ||\underline{\underline{\nabla}}\,\underline{v}||^2_{L^2} + \sum_{K\in\mathcal{T}_h} h_K^2 \left\{||\Delta \underline{v}||^2_{L^2(K)} + ||\underline{\nabla} q||^2_{L^2(K)}\right\} \tag{1.11}$$

always with usual notation, and

$$|(\underline{v},q)|^2_{2,h} = \sum_{K\in\mathcal{T}_h} h_K^2 || - \Delta\underline{v} + \underline{\nabla} q||^2_{L^2(K)} . \tag{1.12}$$

We assume now that we are also given a bilinear form $\mathcal{A}_2(U,V)$ on $\mathcal{V}_2(h)\times\mathcal{V}_2(h)$ such that, for every W,V in $\mathcal{V}_2(h)$

$$\mathcal{A}_2(W,V) \leq M_2\, ||W||_{2,h}\, ||V||_{2,h} \tag{1.13}$$

$$\mathcal{A}_2(V,V) \geq \gamma_2 |V|^2_{2,h} \tag{1.14}$$

with positive constants M_2 and γ_2 independent of h. We set

$$< f_2, V >:= \mathcal{A}_2(U_1,V) \qquad \forall V \in \mathcal{V}_2(h) . \tag{1.15}$$

Note that (1.15) will not produce a reasonable right–hand side, unless it is effectively computable without knowing the solution U_1 of (1.4) explicitely. For instance, for Stokes problem, if $U_1 = (\underline{u},p)$ and $-\Delta\underline{u} + \underline{\nabla} p = \underline{f}$, we can set

$$\mathcal{A}_2(U,V) = \sum_{K\in\mathcal{T}_h} h_K^2 \left(-\Delta\underline{u} + \underline{\nabla} p, -\Delta\underline{v} + \underline{\nabla} q\right)_{L^2(K)} \tag{1.16}$$

so that

$$< f_2, V >= \sum h_K^2 \left(\underline{f}, -\Delta\underline{v} + \underline{\nabla} q\right)_{L^2(K)} \tag{1.17}$$

which is computable without explicit knowledge of the solution U_1.

We can now set $\mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2$ and $f = f_1 + f_2$. It is pretty obvious that the problem

$$\begin{cases} \text{find } U_h \in \mathcal{V}_h \text{ such that} \\ \mathcal{A}\left(U_h, V_h\right) =< f, V_h > \quad \forall V_h \in \mathcal{V}_h \end{cases} \tag{1.18}$$

has a unique solution, since

$$\mathcal{A}(V_h, V_h) \geq \gamma_1 \left|V_h\right|^2 + \gamma_2 |V_h|_{2,h}^2 \geq \gamma\, ||V_h||_{2,h}^2 \tag{1.19}$$

for every V_h in $\mathcal{V}_h$, from (1.7), (1.14) and (1.10).

On the other hand, it is also obvious that

$$\begin{aligned} \mathcal{A}(U_1, V_h) &= \mathcal{A}_1(U_1, V_h) + \mathcal{A}_2(U_1, V_h) \\ &= < f_1, V_h > + < f_2, V_h > = < f, V_h > \end{aligned} \tag{1.20}$$

for every V_h in $\mathcal{V}_h$, using (1.4), (1.15), and the definitions of $\mathcal{A}$ and f. It is still not clear whether U_h converges to U_1 (possibly with optimal rate) or not. For this we need some further assumptions. We assume therefore that there exists a linear operator $\pi_h : \mathcal{V}_1 \to \mathcal{V}_h$ and another seminorm, $|\cdot|_{\star,h}$, on $\mathcal{V}_1$ such that, for every $W \in \mathcal{V}_1$ and $V_h \in \mathcal{V}_h$, we have

$$\mathcal{A}_1\left(W - \pi_h W,\, V_h\right) \;\leq M_l\, |W - \pi_h W|_{\star,h}\; ||V_h||_{2,h} \tag{1.21}$$

$$\mathcal{A}_2\left(V_h, W - \pi_h\, W\right) \;\leq M_r\, |W - \pi_h W|_{\star,h}\; ||V_h||_{2,h} \tag{1.22}$$

$$|W - \pi_h W|_{\star,h} \;\leq M_\star\, ||W||_1 \tag{1.23}$$

$$||\pi_h W||_1 \;\leq M_p ||W||_1 \tag{1.24}$$

with M_l, M_r, $M_\star$, M_p constants independent of h.

In the case of Stokes problems, restricting ourselves, for the sake of simplicity, to finite element approximations with continuous pressures (but the case of discontinuous pressures can also be easily treated, following essentially [13]) we can take as π_h the usual projection (in $\mathcal{V}_1$) onto $\mathcal{V}_h$. It is easy to check that, in this case, for $V = (\underline{v}, q)$ we can take

$$|V|_{\star,h}^2 = ||\underline{\underline{\nabla}}\,\underline{v}||_{L^2}^2 + h^{-2}\; ||\underline{v}||_{L^2}^2 + ||p||_{L^2/\mathbb{R}}^2 \tag{1.25}$$

and (1.21)–(1.24) will hold. Notice also that, for $U_1 = (\underline{u}, p)$ smooth enough, the three quantities

$$||U_1 - \pi_h U_1||_1\;, \quad ||U_1 - \pi_h U_1||_{2,h}\;, \quad |U_1 - \pi_h U_1|_{\star,h} \tag{1.26}$$

have the same order of magnitude (in powers of h), and correspond to optimal orders of accuracy.

We can now prove our two basic results.

Theorem 1.1: There exists a constant δ_1 such that, for all h we have

$$||U_1 - U_h||_{2,h} \leq \delta_1 \left(||U_1 - \pi_h U_1||_{2,h} + |U_1 - \pi_h U_1|_{\star,h} \right) . \tag{1.27}$$

Proof: We have

$$\begin{aligned} &\gamma \, ||U_h - \pi_h U_1||^2_{2,h} \ \leq \qquad\qquad\qquad \text{(use(1.19))} \\ &\leq \mathcal{A}\,(U_h - \pi_h U_1,\, U_h - \pi_h U_1) = \qquad \text{(use(1.18), (1.20))} \\ &= \mathcal{A}\,(U_1 - \pi_h U_1,\, U_h - \pi_h U_1) \ \leq \quad \text{(use(1.21),)} \\ &\leq M_l \, |U_1 - \pi_h U_1|_{\star,h} \ ||U_h - \pi_h U_1||_{2,h} \end{aligned} \tag{1.28}$$

which easily implies

$$||U_h - \pi_h U_1||_{2,h} \ \leq (M_l/\gamma)\, |U_1 - \pi_h U_1|_{\star,h} \tag{1.29}$$

and the result follows by the triangle inequality. ∎

Theorem 1.2: There exists a constant δ_2 such that for all h we have

$$||U_h - U_1||_1 \leq \delta_2 \ \left(||U_1 - \pi_h U_1||_{2,h} + |U_1 - \pi_h U_1|_{\star,h} \right) . \tag{1.30}$$

Proof: We have from (1.3) that there exists V_r in $\mathcal{V}_1$ such that

$$\mathcal{A}_1(U_h - \pi_h U_1,\, V_r) \geq \beta_r \quad ||\, U_h - \pi_h U_1 \,||_1 \quad ||\, V_r \,||_1 \ . \tag{1.31}$$

Then we have

$$\begin{aligned} &\beta_r \ \ ||\, V_r \,||_1 \ \ ||U_h - \pi_h U_1||_1 \leq \mathcal{A}_1\,(U_h - \pi_h U_1,\, V_r) \ = \ (\pm \pi_h V_r) \\ &= \mathcal{A}_1\,(U_h - \pi_h U_1,\, V_r - \pi_h V_r) + \mathcal{A}_1\,(U_h - \pi_h U_1,\, \pi_h V_r) \leq \ \text{(use(1.22), (1.23))} \\ &\leq M_r M_\star \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 + \mathcal{A}_1\,(U_h - \pi_h U_1,\, \pi_h V_r) = (\pm \mathcal{A}_2) \\ &= M_r M_\star \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 + \mathcal{A}\,(U_h - \pi_h U_1,\, \pi_h V_r) - \\ &- \mathcal{A}_2\,(U_h - \pi_h U_1,\, \pi_h V_r) \leq \quad \text{(use(1.18), (1.20) and (1.13))} \\ &\leq M_r M_\star \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 + \mathcal{A}\,(U_1 - \pi_h U_1,\, \pi_h V_r) + \\ &+ M_2 \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, \pi_h \, V_r \,||_{2,h} \leq \ \text{(use } \mathcal{A} = \mathcal{A}_1 + \mathcal{A}_2, \text{(1.21), (1.13))} \\ &\leq M_r M_\star \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 + M_l \ \ |\, U_1 - \pi_h U_1 \,|_{\star,h} \ \ ||\, \pi_h V_r \,||_{2,h} + \\ &+ M_2 \ \ ||\, U_1 - \pi_h U_1 \,||_{2,h} \ \ ||\, \pi_h V_r \,||_{2,h} + M_2 \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, \pi_h V_r \,||_{2,h} \leq \\ &\text{(use (1.9), (1.24))} \ \leq (M_r M_\star + M_2 M_e M_p) \ \ ||\, U_h - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 + \\ &+ M_e M_l M_p \ \ |\, U_1 - \pi_h U_1 \,|_{\star,h} \ \ ||\, V_r \,||_1 + M_e \, M_2 M_p \ \ ||\, U_1 - \pi_h U_1 \,||_{2,h} \ \ ||\, V_r \,||_1 \, , \end{aligned} \tag{1.32}$$

from which

$$\| U_h - \pi_h U_1 \|_1 \le C\big(\|U_h - \pi_h U_1 \|_{2,h} + |U_1 - \Pi_h U_1|_{\star,h} + \|U_1 - \pi_h U_1\|_{2,h}\big) \quad (1.33)$$

and the result follows from (1.29) and (1.33). ∎

Remark 1.2: We notice that, for piecewise polynomial approximations of the Stokes problem, both (1.27) and (1.30) provide error estimates of optimal order (as it was already shown by [9] with a direct approach). In Section 3 we shall apply these abstract results to domain decomposition methods.

2 The continuous formulation for a domain decomposition method

Let us consider, for the sake of simplicity, a polygonal domain $\Omega \subset \mathbb{R}^2$ split into a finite number of polygonal subdomains Ω_k $(k = 1, .., N)$. Let

$$\Omega = \overline{\bigcup_k \Omega_k}^{\circ} \quad ; \quad \Gamma_k = \partial\Omega_k \quad ; \quad \Sigma = \bigcup_k \Gamma_k. \tag{2.1}$$

Let A be a linear elliptic operator of the form

$$Au = \sum_i \left\{ \sum_j (-\frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i} + b_j(x)u) + c_i(x)\frac{\partial u}{\partial x_i} \right\} + d(x)u. \tag{2.2}$$

We assume that the coefficients $a_{ij},\ b_j,\ c_i,\ d$ belong to $L^\infty(\Omega)$ and are smooth in each Ω_k, and we consider the bilinear forms associated with A in each Ω_k, that is,

$$\begin{aligned} &for\ u,\, v \in\ H^1(\Omega_k): \\ &a_k(u, v) := \int_{\Omega_k} \left\{ \sum_i (\sum_j (a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} + b_j u \frac{\partial v}{\partial x_j}) + c_i \frac{\partial u}{\partial x_i} v) + duv \right\} dx. \end{aligned} \tag{2.3}$$

We also set, for $u,\, v \in \prod_k H^1(\Omega_k)$

$$a(u, v) := \sum_k a_k(u, v); \tag{2.4}$$

for the sake of simplicity we also assume that there exists a constant $\alpha > 0$ such that

$$a(v, v) \ge \alpha ||v||^2_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega). \tag{2.5}$$

From now on we are going to use the following notation: $(.,.)$ will be the usual inner product in $L^2(\Omega)$; for $k = 1, ..., N$, $< .,. >_k$ will be the inner product in $L^2(\Gamma_k)$

(or, when necessary, the duality pairing between $H^{-\frac{1}{2}}(\Gamma_k)$ and $H^{\frac{1}{2}}(\Gamma_k)$). Let us now introduce the spaces that will be used in our formulation. For $k = 1, ..., N$ we set

$$\Upsilon_k := H^1(\Omega_k) \qquad ; \qquad M_k := H^{-\frac{1}{2}}(\Gamma_k). \tag{2.6}$$

We then define

$$\Upsilon := \prod_k \Upsilon_k \qquad ; \qquad M := \prod_k M_k, \tag{2.7}$$

and

$$\Phi := \{\varphi \in L^2(\Sigma) \ : \ \exists v \in H^1_0(\Omega) \text{ with } \varphi = v_{|\Sigma}\} \equiv H^1_0(\Omega)_{|\Sigma}, \tag{2.8}$$

with the obvious norms

$$\| v \|^2_{\Upsilon} = \sum_k ||v^k||^2_{H^1(\Omega_k)} \qquad (v \in \Upsilon;\ v = (v^1, ..., v^N)); \tag{2.9}$$

$$\| \mu \|^2_M = \sum_k ||\mu^k||^2_{H^{-\frac{1}{2}}(\Gamma_k)} \qquad (\mu \in M;\ \mu = (\mu^1, ..., \mu^N)); \tag{2.10}$$

$$\| \varphi \|_{\Phi} = \inf\{ \ | v |_{H^1(\Omega)} \mid v \in H^1_0(\Omega),\ v_{|\Sigma} = \varphi\}. \tag{2.11}$$

For every f, say, in $L^2(\Omega)$, we can now consider the following two problems:

$$\begin{cases} \text{find } w \in H^1_0(\Omega) & \text{such that} \\ a(w,v) = (f,v) & \forall v \in H^1_0(\Omega) \end{cases} \tag{2.12}$$

and

$$\begin{cases} \text{find } u \in \Upsilon,\ \lambda \in M \text{ and } \psi \in \Phi \quad \text{such that} \\ \text{i)} \quad a(u,v) - \sum_k < \lambda^k, v^k >_k = \ (f,v) \qquad \forall v \in \Upsilon \\ \text{ii)} \quad \sum_k < \mu^k, \psi - u^k >_k = \ 0 \qquad \forall \mu \in M \\ \text{iii)} \quad \sum_k < \lambda^k, \varphi >_k = 0 \qquad \forall \varphi \in \Phi. \end{cases} \tag{2.13}$$

Theorem 2.1: For every $f \in L^2(\Omega)$, both problems (2.12) and (2.13) have a unique solution. Moreover we have

$$u^k = w \qquad \text{in } \Omega_k \quad (k = 1, ..., N), \tag{2.14}$$

$$\lambda^k = \frac{\partial w}{\partial n^k_A} \qquad \text{on } \Gamma_k \quad (k = 1, ..., N), \tag{2.15}$$

$$\psi = w \qquad \text{on } \Sigma \tag{2.16}$$

where $\partial w / \partial n^k_A$ is the outward conormal derivative (of the restriction of w to Ω_k) with respect to the operator A.

Proof It follows from (2.5) that (2.12) has a unique solution w. Setting u, λ, ψ as in (2.14)–(2.16) it is easy to verify that this is a solution of (2.13). Hence, we only need to show that (2.13) cannot have two different solutions or, in other words, that $f = 0$ in (2.13) implies $u = 0,\ \lambda = 0,\ \psi = 0$. Let then $f = 0$; from (2.13;ii) we get $u^k = \psi$ on Γ_k for every k, and therefore the existence of a function $w \in H_0^1(\Omega)$ such that $\psi = w_{|\Sigma}$ and $u^k = w_{|\Omega_k}$. From (2.13;i) with $v = w$, and (2.13;iii) with $\varphi = w$ we have

$$a(w, w) = 0 \tag{2.17}$$

yielding $u = 0$ and $\psi = 0$. From (2.13;i) we have now

$$< \lambda^k, v >_k = 0 \qquad \forall v \in \Upsilon_k \qquad \forall k, \tag{2.18}$$

which easily gives $\lambda = 0$. ∎

It is very important, for applications to domain decomposition methods, to remark explicitly that the first two equations of (2.13) can be written as

$$\begin{cases} a_k(u^k, v^k) - < \lambda^k, v^k >_k = (f, v^k) & \forall v^k \in \Upsilon_k, \quad \forall k \\ < \mu^k, u^k >_k = < \psi, \mu^k >_k & \forall \mu^k \in M_k, \quad \forall k. \end{cases} \tag{2.19}$$

In particular, for all fixed k, assuming f and ψ as data, (2.19) is the variational formulation of the Dirichlet problem

$$\begin{cases} Au^k = f & \text{in } \Omega_k, \\ u^k = \psi & \text{on } \Gamma_k, \end{cases} \tag{2.20}$$

where the boundary condition is imposed by means of a Lagrange multiplier (that finally comes out to be $\lambda^k \equiv \partial u^k / \partial n_A^k$) as in Babuška [1]. Hence, for f and ψ given, the resolution of the first two equations of (2.13) amounts to the resolution of N independent Dirichlet problems.

3 Regularization of domain decomposition methods

We apply now the abstract technique of Section 1 to problem (2.13). For this, we have first to set it in the form (1.4). We define

$$\mathcal{V}_1 = \Upsilon \times M \times \Phi, \tag{3.1}$$

$$\begin{aligned} \mathcal{A}_1((u, \lambda, \psi), (v, \mu, \varphi)) &= a(u, v) - \sum_k < \lambda^k, v^k >_k \\ &+ \sum_k < \lambda^k, \varphi >_k + \sum_k < \mu^k, u^k >_k - \sum_k < \mu^k, \psi >_k \ . \end{aligned} \tag{3.2}$$

One can easily check that (1.1)-(1.3) hold true. Non–negativity (1.7) is trivial if we take

$$|(v,\mu,\varphi)|^2 := \sum_k \|\underline{\nabla} v\|^2_{L^2(\Omega_k)} \tag{3.3}$$

(using (2.5)). Assume now that we are given finite dimensional subspaces $\mathcal{V}_h = \Upsilon_h \times M_h \times \Phi_h$. For the sake of simplicity we may think that we have a global decomposition $\mathcal{T}_h$ of Ω into finite elements ω (say, triangles), which is compatible with the macro-element subdivision (2.1) (in other words, for every ω in $\mathcal{T}_h$ and for every Ω_k, the symmetric difference $(\Omega_k \cup \omega)\setminus(\Omega_k \cap \omega)$ has zero measure). The decomposition $\mathcal{T}_h$ induces then, in a natural way, finite element decompositions of each Ω_k, of each Γ_k, and of Σ. For the sake of simplicity we shall write $\sum_{\omega(k)}$ and $\sum_{\sigma(k)}$ for the sum over those elements ω (resp. σ) belonging to Ω_k (resp. Γ_k). We shall also denote by h_ω and h_σ the diameter of ω and σ, respectively. As far as the degrees of the polynomials are concerned, we allow the maximum generality; the degree can also change from one macro-element to another. Since $\mathcal{V}_h \subset \mathcal{V}$, the functions v_h^k must be continuous in Ω_k, and φ_h must also be continuous on Σ.

Remark 3.1: Our assumptions on $\mathcal{V}_h$ are much more restrictive than necessary. In principle we can easily adapt these ideas to more general subspaces, even allowing different Galerkin methods (Fourier, spectral, wavelets etc.) from one Ω_k to another. However, as we shall see, the notation (more than the actual implementation) is already cumbersome in our simplified case, and would become too heavy in a more general one. ■

From now on we shall often write U and V instead of (u,λ,ψ) and (v,μ,φ).

The bilinear form $\mathcal{A}_2(U,V)$ that we want to add to $\mathcal{A}_1(U,V)$ is the Fréchet derivative of the functional

$$\begin{aligned} J(V) \;=\; (1/2)|V|^2_{2,h} = \frac{1}{2}\sum_k \Big\{ \sum_{\omega(k)} h_\omega^2 \, \| A v^k \|^2_{L^2(\omega)} + \\ + \sum_{\sigma(k)} (h_\sigma \, \| \mu^k - \partial v^k/\partial n_A^k \|^2_{L^2(\sigma)} + h_\sigma \, \| v^k - \varphi \|^2_{H^1(\sigma)}) \Big\}, \end{aligned} \tag{3.4}$$

(i.e., $\mathcal{A}_2(U,V)$ is the bilinear symmetric form associated with (3.4)). Note that (3.4) also defines the seminorm $|\cdot|_{2,h}$. The norm $\|\cdot\|_{2.h}$ can now be defined in a natural way as

$$\begin{aligned} \| V \|^2_{2,h} = \; | V |^2 + \sum_k \Big\{ \sum_{\omega(k)} h_\omega^2 \, \| A v^k \|^2_{L^2(\omega)} + \\ + \sum_{\sigma(k)} (h_\sigma \, \| \mu^k \|^2_{L^2(\sigma)} + h_\sigma \, \| \partial v^k/\partial n_A^k \|^2_{L^2(\sigma)} + h_\sigma \, \| \varphi \|^2_{H^1(\sigma)}) \Big\}. \end{aligned} \tag{3.5}$$

The space $\mathcal{V}_2(h)$ will be defined accordingly (i.e., as the set of the V's in $\mathcal{V}_1$ such that (3.5) is finite). Note that (1.13)-(1.14) are trivial, while (1.9)-(1.10) can be proved (with arguments very similar to those in [9] and [3], for instance) by means of local inverse inequalities for piecewise polynomials. Note also that, if U_1 is the exact solution of (2.13), we easily have

$$\mathcal{A}_2(U_1, V) = \sum_k \sum_{\omega(k)} h_\omega^2 (f, Av^k)_{L^2(\omega)} \tag{3.6}$$

which is easily computable.

We can now write problem (1.18), in the present application, in its expanded form:

find $(u_h,\ \lambda_h,\ \psi_h) \in \Upsilon_h \times M_h \times \Phi_h$ such that

$$\sum_k \{a_k(u_h^k, v_h^k) - < \lambda_h^k, v_h^k >_k + \sum_{\omega(k)} h_\omega^2 (Au_h^k, Av_h^k)_{L^2(\omega)} + \tag{3.7}$$

$$- \sum_{\sigma(k)} h_\sigma [(\lambda_h^k - \partial u_h^k/\partial n_A^k, \partial v_h^k/\partial n_A^k)_{L^2(\sigma)} + (u_h^k - \psi_h, v_h^k)_{H^1(\sigma)}]\} =$$

$$= \sum_k \sum_{\omega(k)} (f, v_h^k + Av_h^k)_{L^2(\omega)} \qquad \forall v_h \in \Upsilon_h$$

$$\sum_k \{< \mu_h^k, u_h^k - \psi_h >_k + \sum_{\sigma(k)} h_\sigma (\lambda_h^k - \partial u_h^k/\partial n_A^k, \mu_h^k)_{L^2(\sigma)}\} = 0 \quad \forall \mu_h \in M_h \tag{3.8}$$

$$\sum_k \left\{ < \lambda_h^k, \varphi_h >_k + \sum_{\sigma(k)} h_\sigma (u_h^k - \psi_h, \varphi_h)_{H^1(\sigma)} \right\} = 0 \qquad \forall \varphi_h \in \Phi_h. \tag{3.9}$$

Problem (3.7) – (3.9) has clearly a unique solution. We point out explicitly that the regularized formulation (3.7) – (3.9) is still well suited for parallel implementation. Indeed, for ψ_h and f given, the resolution of (3.7) – (3.8) amounts to the resolution of N independent problems, each of them being a Dirichlet problem with Lagrange multipliers treated with a variant of [3]. For studying the convergence of (3.7) – (3.9) we have to introduce a $|\cdot|_{\star,h}$ norm. An easy computation shows that, by setting

$$|V|_{\star,h}^2 = \|v\|_\Upsilon^2 + \sum_k \sum_{\sigma(k)} h_\sigma^{-1} \left(\|v^k\|_{L^2(\sigma)}^2 + \|\varphi\|_{L^2(\sigma)}^2 + \|\mu^k\|_{(H^1(\sigma))'}^2 \right) \tag{3.10}$$

(where $\left(H^1(\sigma)\right)'$ is the dual space of $H^1(\sigma)$), properties (1.21) and (1.22) hold for any reasonable choice of $\pi_h : \mathcal{V}_1 \to \mathcal{V}_h$. Hence, we only need to choose π_h in such a way that (1.23), (1.24) hold and both $\|U_1 - \pi_h U_1\|_{2,h}$ and $|U_1 - \pi_h|_{\star,h}$ provide estimates of optimal order. This can be done very easily in many ways. Let us see, for instance, what can then be deduced from, say, Theorem 1.2 as an estimate for the error. To fix the ideas, assume that we approximate u, λ, ψ with piecewise polynomials of degree r, s, t respectively. Clearly $r \geq 1,\ s \geq 0,\ t \geq 1$. If the solution $(u,\ \lambda,\ \psi)$ is smooth enough we have the following estimate

$$\| \psi - \psi_h \|^2_{H^{1/2}(\Sigma)} + \sum_k \Big(\| u^k - u^k_h \|^2_{H^1(\Omega_k)} + \| \lambda^k - \lambda^k_h \|^2_{H^{-1/2}(\Gamma_k)} \Big) \leq \tag{3.11}$$

$$\leq \delta \sum_k \Big\{ \sum_{\omega(k)} h^{2r}_\omega \| u^k \|^2_{H^{r+1}(\omega)} + \sum_{\sigma(k)} \big(h^{2s+3}_\sigma \| \lambda^k \|^2_{H^{s+1}(\sigma)} + $$

$$+ h^{2t+1}_\sigma \| \psi \|^2_{H^{t+1}(\sigma)} \big) \Big\}$$

with δ constant independent of h.

References

1. I. Babuška: The finite element method with lagrangian multipliers, Numer. Math., 20, 179–192, 1973.
2. C. Baiocchi and F. Brezzi: in preparation.
3. H.J.C. Barbosa, T.J.R. Hughes: Boundary Lagrange multipliers in finite element methods: error analysis in natural norms, Numer. Math., 62, 1–16, 1992.
4. C. Bernardi, Y. Maday and A.T. Patera: A new nonconforming approach to domain decompositions: The mortar element method. In *Nonlinear Partial Differential Equations and their Applications* (H.Brezis-J.L.Lions eds.), Pitman and Wiley, 1989.
5. F. Brezzi and M. Fortin: *Mixed and hybrid finite element methods*, Springer Verlag, New York 1991.
6. F. Brezzi and L.D. Marini: Macro hybrid elements and domain decomposition methods. (To appear in *Proc. Colloque en l'honneur du 60ème anniversaire de Jean Cea*, Sophia-Antipolis, April 1992).
7. F. Brezzi and L.D. Marini: A three-field domain decomposition method. (To appear in *Proc. 6th International Conference on Domain Decomposition Methods in Science and Engineering*, Como, June 1992).
8. F. Brezzi and L.D. Marini: in preparation.
9. J. Douglas, jr. and J. Wang: An absolutely stabilized finite element method for the Stokes problem, Math. Comp., 52, 495–508, 1989.
10. L.P. Franca and S.L. Frey: Stabilized finite element methods: II. The incompressible Navier-Stokes equations, (to appear in Comput. Meths. Appl. Mech. Engrg.).
11. L.P. Franca, S.L. Frey and T.J.R. Hughes: Stabilized finite element methods: I. Application to the advective- diffusive model, Comput. Meths. Appl. Mech. Engrg., 95, 253–276, 1992.
12. L.P. Franca and T.J.R. Hughes: Two classes of mixed finite element methods, Comput. Meths. Appl. Mech. Engrg., 69, 89–129, 1988.
13. L.P. Franca and R. Stenberg: Error analysis of some Galerkin least square methods for the elasticity equations, SIAM J. Numer. Anal., 28, 1680–1697, 1991.