VIRTUAL ELEMENT AND DISCONTINUOUS GALERKIN METHODS

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Abstract. Virtual Element Methods (VEM) are the latest evolution of the Mimetic Finite Difference Method, and can be considered to be more close to the Finite Element approach. They combine the ductility of mimetic finite differences for dealing with rather weird element geometries with the simplicity of implementation of Finite Elements. Moreover, they make it possible to construct quite easily high-order and high-regularity approximations (and in this respect they represent a significant improvement with respect to both FE and MFD methods). In the present paper we show that, on the other hand, they can also be used to construct DG-type approximations, although numerical tests should be done to compare the behavior of DG-VEM versus DG-FEM.

Key words. Discontinuous Galerkin, Virtual Elements, Mimetic Finite Differences

AMS(MOS) subject classifications. 65N30, 65N12, 65G99, 76R99

1. Introduction. The aim of this paper is to present a possible way to introduce the Virtual Element Method (VEM) in the Discontinuous Galerkin (DG) framework. From several points of view VEM can be considered as the natural extension of Finite Element Methods to more general geometries and continuity requirements. Apparently, their extension to the Discontinuous Galerkin world could be seen as useless, as DG methods can already deal with rather general geometries. However, in a certain number of their applications there is some need of a conforming interpolant, that for general geometries or for higher order continuity (as for plate problems, among others) will not be easily available within the usual DG framework. Here, however, to start with, we will deal with the simplest possible case, that is the discretization of the Poisson problem in two dimensions. The idea is to start understanding what are the most convenient ways to deal with Discontinuous Virtual Elements. We shall see that a direct application of the DG technology cannot be done, but some simple variants are available that still ensure uniqueness, stability, and convergence with optimal error bounds.

As a first step we will recall the basic concepts of Virtual Element Methods. This will be done with some details, taking into account that the introduction of VEM is quite recent, and we cannot expect many readers to be familiar with them. In the next section we will present the basic assumptions (on the element geometry, on the discrete spaces) and recall an abstract convergence result. Then we will recall the general way to construct the discrete bilinear form, in Section 3, and the discrete right-

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hand side, in Section 4. In Section 5 we will recall the classical instruments and concepts of DG formulations (in a much less detailed way, this time). The novelty of the paper will appear in Section 6, where VEM will be adapted to DG formulations, and in Section 7 where optimal error bounds will be proved.

Throughout the paper, we will follow the usual notation for Sobolev spaces and norms (see e.g. [6]). In particular, for an open bounded domain $\mathcal{D}$, we will use $|\cdot|_{s,\mathcal{D}}$ and $\|\cdot\|_{s,\mathcal{D}}$ to denote seminorm and norm, respectively, in the Sobolev space $H^s(\mathcal{D})$, while $(\cdot,\cdot)_{0,\mathcal{D}}$ will denote the $L^2(\mathcal{D})$ inner product. Often the subscript will be omitted when $\mathcal{D}$ is the computational domain Ω. For a nonnegative integer k, the space of polynomials of degree less than or equal to k will be denoted by $\mathbb{P}_k$. Following a common convention, we will also use $\mathbb{P}_{-1} := \{0\}$.

Finally, C will be a generic constant independent of the decomposition that could change from an occurrence to the other.

2. Basic Assumptions and an abstract convergence result. We first recall the general idea of continuous Virtual Element Methods, underlying the similarities with classical Finite Element Methods (we refer to [3] for a more detailed presentation).

For this we consider, as usual, the simplest possible problem: *find* $u \in V \equiv H^1_0(\Omega)$ *such that* $-\Delta u = f$. Written in variational form, the problem becomes

$$\textit{find } u \in V \equiv H^1_0(\Omega) \textit{ such that } a(u,v) = (f,v) \quad \forall v \in V, \tag{2.1}$$

where (as usual):

$$a(u,v) := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}\mathbf{x}, \qquad (f,v) = \int_\Omega f\, v\, dx. \tag{2.2}$$

Let $\mathcal{T}_h$ be a decomposition of Ω into polygons of almost arbitrary shape (see, as an example, Fig. 1). On $\mathcal{T}_h$ we make the following assumptions

H1 - There exists an integer N and a positive real number ζ such that for every h and for every $K \in \mathcal{T}_h$:

- the number of edges of K is $\le N$,
- the ratio between the shortest edge and the diameter h_K of K is bigger than ζ, and
- K is star-shaped with respect to every point of a ball of radius ζh_K.

□

REMARK 2.1. *We point out that from assumption* **H1** *we can easily deduce that there exists an* $s* > 3/2$*, depending on* ζ*, such that for every smooth* g *on* ∂K *and for every smooth* f *in* K *the solution* φ *of the problem* $\Delta\varphi = f$ *in* K *with* $\varphi = g$ *on* ∂K *belongs to* $H^{s*}(K)$.

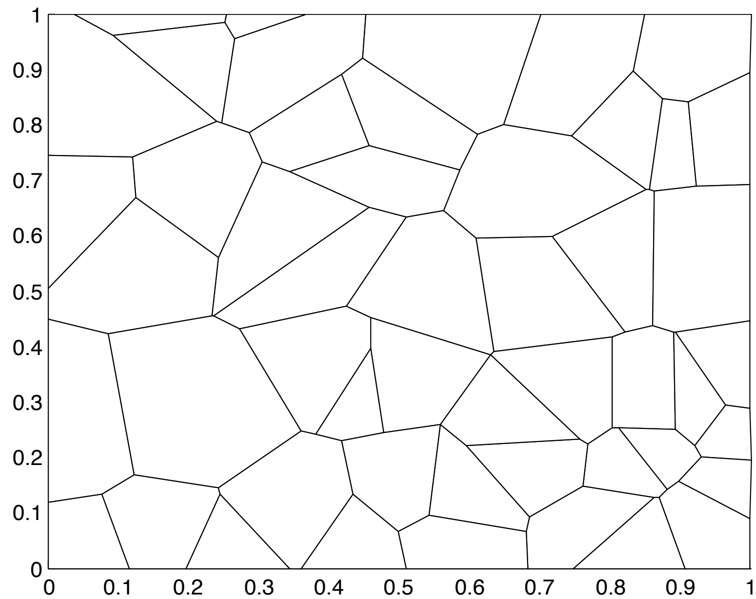

FIG. 1. *Example of a Voronoi tessellation*

Next, we fix an integer $k \geq 1$ (that will be our order of accuracy) and define for each $K \in \mathcal{T}_h$

$$V_k^K := \{v : v_{|e} \in \mathbb{P}_k(e) \ \forall \text{ edge } e \text{ of } K, \Delta v \in \mathbb{P}_{k-2}(K)\}, \tag{2.3}$$

where we recall that $\mathbb{P}_k$ denotes the space of polynomials of degree $\leq k$, and $\mathbb{P}_{-1} := \{0\}$.

Denoting by NV the number of vertices of K (obviously equal, as well, to the number of edges), the dimension of V_k^K will clearly be

$$N^K := NV + NV * (k-1) + \frac{k(k-1)}{2} = NV * k + \frac{k(k-1)}{2}$$

An element v of V_k^K can be identified by

- a) the values of v at the vertices;
- b) the moments $\int_e v\, p_{k-2}\, \mathrm{d}s$ on each edge e, $\quad k \geq 2$;
- c) the moments $\int_K v\, p_{k-2}\, \mathrm{d}\mathbf{x}$, $\quad k \geq 2$.

THEOREM 2.1. *For every $k \geq 1$ the set of degrees of freedom a), b), c) are unisolvent for the space V_k^K.*

Proof. The number of degrees of freedom a), b), c) equals the dimension of V_k^K. Hence we have only to check that every $v \in V_k^K$ having the

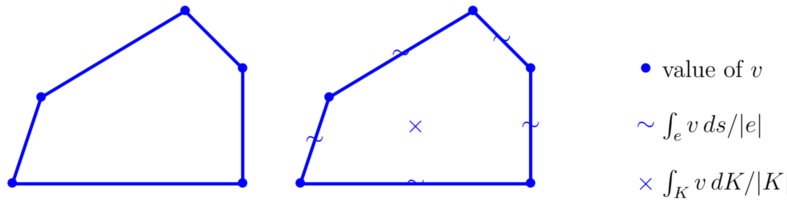

FIG. 2. *Example of d.o.f. for $k = 1$ (left), and $k = 2$ (right)*

d.o.f.'s equal to zero is identically zero. For this we first observe that if the degrees of freedom a) and b) are equal to zero, then $v = 0$ on ∂K. Remember that v, being in V_k^K, has Δv in $\mathbb{P}_{k-2}$. Hence, if the d.o.f. c) are equal to 0, we have

$$0 = \int_K (-\Delta v) v \,\mathrm{d}\mathbf{x} = |v|^2_{1,K}$$

implying that $v \equiv 0$. □

For later use, it will be however more convenient to define the degrees of freedom in a more precise way. For this, for a geometric object $\mathcal{O} \subset \mathbb{R}^d$ (as an edge, a face, a d-dimensional domain, etc.) we define first its barycenter $\mathbf{x}_{\mathcal{O}}$ and its diameter $d_{\mathcal{O}}$. Then we consider, for every integer $r \geq 0$, the set $\mathcal{M}_r(\mathcal{O})$ of all *monomials*, in $\mathbb{R}^d$, of the type

$$\mathcal{M}_r(\mathcal{O}) := \left\{ \frac{(\mathbf{x} - \mathbf{x}_{\mathcal{O}})^{\alpha}}{d_{\mathcal{O}}^{|\alpha|}} \right\} \qquad \text{for } |\alpha| \leq r \tag{2.4}$$

where for the multi-integer $\alpha \in \mathbb{N}^d$ we followed the usual notation

$$(x_1, ..., x_d)^{\alpha} \equiv x_1^{\alpha_1} \cdot x_2^{\alpha_2} \cdots x_d^{\alpha_d} \qquad \text{and} \qquad |\alpha| = \sum_{i=1}^{d} \alpha_i.$$

Now we can make precise the actual degrees of freedom that we want to use in V_k^K:

- the values of v at the vertices;

  and for $k \geq 2$
- the moments $\displaystyle\int_e v\, m_{k-2}\,\mathrm{d}s/|e|$, $\quad m_{k-2} \in \mathcal{M}_{k-2}(e)$, on each edge e,
- the moments $\displaystyle\int_K v\, m_{k-2}\,\mathrm{d}\mathbf{x}/|K|$, $\quad m_{k-2} \in \mathcal{M}_{k-2}(K)$.

Fig. 2 shows an example of d.o.f for the cases $k = 1$ and $k = 2$.

For each h and for each k we then define the VEM space as

$$V_h := \{v \in V : \; v_{|K} \in V_k^K \; \forall K \in \mathcal{T}_h\}. \tag{2.5}$$

Following [3], we need now to define an element $f_h \in V_h'$, and a bilinear form $a_h(\cdot,\cdot)$ from $V_h \times V_h$ to $\mathbb{R}$ satisfying the following assumptions:

H2 • k-consistency: for all h, and for all K in $\mathcal{T}_h$

$$\forall p \in \mathbb{P}_k, \, \forall v_h \in V_h \qquad a_h^K(p, v_h) = a^K(p, v_h). \tag{2.6}$$

- Stability: $\exists$ two positive constants α_* and α^*, independent of h and of K, such that

$$\forall v_h \in V_h \quad \alpha_* \, a^K(v_h, v_h) \le a_h^K(v_h, v_h) \le \alpha^* \, a^K(v_h, v_h). \tag{2.7}$$

□

In (2.6)–(2.7) $a^K(\cdot,\cdot)$ denotes the restriction of the bilinear form $a(\cdot,\cdot)$ defined in (2.2) to the element K. We point out that, due to the symmetry of $a(\cdot,\cdot)$, (2.7) implies as well continuity:

$$\begin{aligned} a_h^K(v_h, w_h) &\le \Big(a_h^K(v_h, v_h)\Big)^{1/2} \Big(a_h^K(w_h, w_h)\Big)^{1/2} \\ &\le \alpha^* \, (a^K(v_h, v_h))^{1/2} (a^K(w_h, w_h))^{1/2} \le \alpha^* \, |v_h|_{1,K} \, |w_h|_{1,K}. \end{aligned} \tag{2.8}$$

Then, the approximate problem is, as usual,

$$\textit{find } u_h \in V_h \textit{ such that} \quad a_h(u_h, v_h) = < f_h, v_h > \quad \forall v_h \in V_h. \tag{2.9}$$

The following convergence result is proved in [3].

THEOREM 2.2. *Under Assumptions* **H2**, *the discrete problem* (2.9) *has a unique solution u_h. Moreover, for every approximation u_I of u in V_h, and for every approximation u_π of u that is piecewise in $\mathbb{P}_k$, we have*

$$\|u - u_h\|_V \; \le C\Big(\|u - u_I\|_V + \|u - u_\pi\|_{h,V} + \|f - f_h\|_{V_h'}\Big)$$

where C is a constant independent of h and

$$\|f - f_h\|_{V_h'} := \sup_{v_h \in V_h} \frac{< f - f_h, v_h >}{\|v_h\|_V}.$$

3. Construction of the bilinear form $a_h(u_h, v_h)$. First of all, we observe that the local degrees of freedom allow us to compute exactly $a^K(p, v)$ for any $p \in \mathbb{P}_k(K)$ and for any $v \in V_k^K$. Indeed, observe first that the value of each function $v \in V_h$ at the boundary of each element is known (it is a polynomial!), even when the value inside the element is not. Then consider the following integration by parts

$$a^K(p,v) = \int_K \nabla p \cdot \nabla v dx = -\int_K \Delta p \, v dx + \int_{\partial K} \frac{\partial p}{\partial n} v ds, \qquad (3.1)$$

and observe that since $\Delta p \in \mathbb{P}_{k-2}(K)$ and $\partial p/\partial n \in \mathbb{P}_{k-1}(e)$ for all $e \subset \partial K$, the last two integrals can be computed exactly knowing only the degrees of freedom associated with v (and without necessarily knowing v in the interior of K).

This allows us to define (and compute!) the (projection) operator $\Pi_k^K : V_k^K \longrightarrow \mathbb{P}_k(K) \subset V_k^K$ as follows: for all $v \in V_k^K$ we define $\Pi_k^K v$ as the solution of

$$\begin{cases} (\nabla \Pi_k^K v, \nabla q)_{0,K} = (\nabla v, \nabla q)_{0,K} \quad \forall q \in \mathbb{P}_k(K) \\ \displaystyle\int_{\partial K} \Pi_k^K v \, \mathrm{d}s = \int_{\partial K} v \, \mathrm{d}s. \end{cases} \qquad (3.2)$$

We note that (3.2) clearly implies

$$\Pi_k^K q = q, \quad \forall q \in \mathbb{P}_k(K), \qquad (3.3)$$

since the first equation in (3.2) tells us that q and $\Pi_k^K q$ have the same gradient, and the second equation takes care of the constant part.

At this point, we observe that choosing $a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v)$ would easily ensure property (2.6). However this choice would not, in general, satisfy (2.7). Therefore we need to add a term able to ensure (2.7). Let then $S^K(u,v)$ be a symmetric positive definite bilinear form (to be chosen) that verifies

$$c_* a^K(v,v) \le S^K(v,v) \le c^* a^K(v,v) \quad \forall v \in V_k^K \quad \text{with } \Pi_k^K v = 0 \qquad (3.4)$$

for some positive constants $c_*,\ c^*$ independent of K and h_K. Then we set

$$a_h^K(u,v) = a^K(\Pi_k^K u, \Pi_k^K v) + S^K(u - \Pi_k^K u, v - \Pi_k^K v) \quad \forall u, v \in V_k^K. \quad (3.5)$$

THEOREM 3.1. *The bilinear form* (3.5) *satisfies the consistency property* (2.6) *and the stability property* (2.7).

Proof. Property (2.6) follows immediately from (3.3) and (3.2): for $p \in \mathbb{P}_k(K)$ (3.3) implies $S^K(p - \Pi_k^K p, v - \Pi_k^K v) = 0$. Hence, for all $v \in V_k^K$, using (3.5) and (3.2), we have

$$a_h^K(p,v) = a^K(\Pi_k^K p, \Pi_k^K v) = a^K(p,v). \qquad (3.6)$$

Then we observe first that, since $a_h^K(v - \Pi_k^K v, \Pi_k^K v) \equiv 0$ for all v, we easily have

$$a_h^K(v,v) = a_h^K(\Pi_k^K v, \Pi_k^K v) + a_h^K(v - \Pi_k^K v, v - \Pi_k^K v) \quad \forall v \in V_k^K. \quad (3.7)$$

Property (2.7) now follows from (3.4) and (3.7) with $\alpha^* := \max\{1, c^*\}$ and $\alpha_* := \min\{1, c_*\}$: indeed for all $v \in V_k^K$

$$\begin{aligned} a_h^K(v,v) &\leq a^K(\Pi_k^K v, \Pi_k^K v) + c^* a^K(v - \Pi_k^K v, v - \Pi_k^K v) \\ &\leq \max\{1, c^*\}\Big(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v)\Big) \\ &= \alpha^* a^K(v,v), \end{aligned}$$

and similarly

$$\begin{aligned} a_h^K(v,v) &\geq \min\{1, c_*\}\Big(a^K(\Pi_k^K v, \Pi_k^K v) + a^K(v - \Pi_k^K v, v - \Pi_k^K v)\Big) \\ &= \alpha_* a^K(v,v). \end{aligned}$$

□

3.1. Choice of S^K. In general, the choice of the bilinear form S^K would depend on the problem and on the degrees of freedom. From (3.4) it is clear that S^K must scale like $a^K(\cdot,\cdot)$ on the kernel of Π_k^K. Denoting by $\chi_i,\ i = 1, ..., N^K$ the i^{th} d.o.f. in V_k^K, and choosing then the canonical basis $\varphi_1, ..., \varphi_{N^K}$ as

$$\chi_i(\varphi_j) = \delta_{ij}, \quad i, j = 1, 2, \ldots, \mathrm{N}^K, \tag{3.8}$$

the local stiffness matrix is given by

$$a_h^K(\varphi_i, \varphi_j) = a^K(\Pi_k^K \varphi_i, \Pi_k^K \varphi_j) + S^K(\varphi_i - \Pi_k^K \varphi_i, \varphi_j - \Pi_k^K \varphi_j). \tag{3.9}$$

In the present case it is easy to check that, on a “reasonable” polygon (like, for instance, the ones that satisfy assumptions **H1**) we have $a^K(\varphi_i, \varphi_i) \simeq 1$. Hence, a possible choice for S^K is simply

$$S^K(\varphi_i - \Pi_k^K \varphi_i, \varphi_j - \Pi_k^K \varphi_j) = \sum_{r=1}^{N^K} \chi_r(\varphi_i - \Pi_k^K \varphi_i)\chi_r(\varphi_j - \Pi_k^K \varphi_j). \tag{3.10}$$

REMARK 3.1. *This explains why, in defining the d.o.f. in V_k^K, we used, instead of the more usual $\mathbb{P}_k$, the set $\mathcal{M}_k$. With the latter choice all the d.o.f. scale like* 1*, and this allows to choose S^K as simple as in* (3.10).

4. Construction of the right-hand side. We consider first the case $k \geq 2$, and define f_h on each element K as the $L^2(K)-$projection of f onto the space $\mathbb{P}_{k-2}$, that is,

$$f_h = P_{k-2}^K f \qquad \text{on each } K \in \mathcal{T}_h.$$

Consequently, the associated right-hand side

$$\begin{aligned} < f_h, v_h > &= \sum_{K \in \mathcal{T}_h} \int_K f_h\, v_h \,\mathrm{d}\mathbf{x} \equiv \sum_{K \in \mathcal{T}_h} \int_K (P_{k-2}^K f)\, v_h \,\mathrm{d}\mathbf{x} \\ &= \sum_{K \in \mathcal{T}_h} \int_K f\,(P_{k-2}^K v_h) \,\mathrm{d}\mathbf{x} \end{aligned}$$

can be exactly computed using the degrees of freedom for V_h that represent the internal moments. Then, standard L^2-orthogonality and approximation estimates on star-shaped domains yield

$$\begin{aligned}< f_h, v_h > -(f, v_h) &= \sum_{K\in\mathcal{T}_h} \int_K (P^K_{k-2} f - f)\, v_h \ \mathrm{d}\mathbf{x} \\ &= \sum_{K\in\mathcal{T}_h} \int_K (P^K_{k-2} f - f)(v_h - P^K_0 v_h)\,\mathrm{d}\mathbf{x} \\ &\le C \sum_{K\in\mathcal{T}_h} h_K^{k-1} |f|_{k-1,K}\ h_K |v_h|_{1,K} \\ &\le C\, h^k (\sum_{K\in\mathcal{T}_h} |f|^2_{k-1,K})^{1/2}\, |v_h|_1,\end{aligned} \tag{4.1}$$

and thus,

$$\|f - f_h\|_{V'} \le C h^k \Big(\sum_{K\in\mathcal{T}_h} |f|^2_{k-1,K} \Big)^{1/2}. \tag{4.2}$$

For the case $k = 1$ we can first, on each element K, define $\overline{v_h}$ as

$$\overline{v_h} := \frac{1}{|\partial K|} \int_{\partial K} v_h \,\mathrm{d}s$$

and then define

$$< f_h, v_h >:= \sum_{K\in\mathcal{T}_h} \int_K f\, \overline{v_h}\,\mathrm{d}\mathbf{x}$$

to obtain

$$< f_h, v_h > - (f, v_h) = \sum_{K\in\mathcal{T}_h} (f, \overline{v_h} - v_h)_{0,K} \le C\, h \|f\|_{0,\Omega}\, |v_h|_{1,\Omega}.$$

5. Basic concepts of DG methods. The extension of what we have presented in the previous sections to DG is almost straightforward. The first difference is, obviously, in the definition of the space V_h, which is now made of discontinuous functions. Let V_{DG} be such a space:

$$V_{DG} := \{v \in L^2(\Omega) :\ v_{|K} \in V^K_k\ \forall K \in \mathcal{T}_h\}, \tag{5.1}$$

where the local spaces V^K_k are still defined as in (2.3). We recall the definition of jumps and averages for scalar and vector-valued functions ($v, \boldsymbol{\tau}$, respectively) on an edge e common to two elements $K_1,\ K_2$ (see [2]):

$$\{v\} = \frac{v^1 + v^2}{2}, \quad [\![\, v \,]\!] = v^1\mathbf{n^1} + v^2\mathbf{n^2}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2}{2}, \quad [\![\, \boldsymbol{\tau} \,]\!] = \boldsymbol{\tau}^1 \cdot \mathbf{n^1} + \boldsymbol{\tau}^2 \cdot \mathbf{n^2}$$

where $\mathbf{n}^1,\ \mathbf{n}^2$ are the outward normal unit vectors to $K_1,\ K_2$. On a boundary edge we only need $[\![v]\!] = v\mathbf{n}$ and $\{\boldsymbol{\tau}\} = \boldsymbol{\tau}$.

We also define, for every $t \geq 0$, the space $H^t(\mathcal{T}_h) := \prod_K H^t(K)$ of piecewise regular functions. We recall from Remark 2.1 that

$$V_{DG} \subset H^{s*}(\mathcal{T}_h) \tag{5.2}$$

for some $s* > 3/2$ depending on the value of ζ. Then for $v,\, w \in H^{s*}(\mathcal{T}_h)$ we set

$$(\nabla v, \nabla w)_h = \sum_K \int_K \nabla v \cdot \nabla w \,\mathrm{d}\mathbf{x}, \quad <\{\nabla v\}, [\![w]\!]>= \sum_e \int_e \{\nabla v\} \cdot [\![w]\!] \,\mathrm{d}s$$

$$<[\![v]\!], [\![w]\!]>= \sum_e \frac{1}{h_e} \int_e [\![v]\!] \cdot [\![w]\!] \,\mathrm{d}s, \quad \|[\![v]\!]\|^2_{0,\partial K} = \sum_{e\subset\partial K} \frac{1}{h_e} \int_e |[\![v]\!]|^2 \,\mathrm{d}s.$$

For $v \in H^2(\mathcal{T}_h)$ we define

$$\|v\|^2_{2,DG} = \sum_{K\in\mathcal{T}_h} \Big(\|\nabla v\|^2_{0,K} + h_K^2 |\nabla v|^2_{1,K} \Big) + <[\![v]\!], [\![v]\!]> . \tag{5.3}$$

We remark that, for functions v_h that are piecewise polynomials, by the usual inverse inequality we have

$$\|v_h\|^2_{2,DG} \simeq \|v_h\|^2_{1,DG} := \sum_{K\in\mathcal{T}_h} \|\nabla v_h\|^2_{0,K} + <[\![v_h]\!], [\![v_h]\!]> . \tag{5.4}$$

We also set, for functions $v,\, w \in H^1(\mathcal{T}_h)$

$$\widetilde{a}(v, w) := \sum_{K\in\mathcal{T}_h} a^K(v, w) = \sum_{K\in\mathcal{T}_h} \int_K \nabla v \cdot \nabla w \,\mathrm{d}\mathbf{x}.$$

We observe that the solution u of (2.1) verifies $[\![\nabla u]\!] = 0$ so that, integrating by parts on each element and recalling that $f = -\Delta u$, we have

$$\widetilde{a}(u, v) - <\{\nabla u\}, [\![v]\!]>= (f, v) \quad \forall v \in V_{DG}. \tag{5.5}$$

On the other hand, the solution u of (2.1) obviously verifies $[\![u]\!] = 0$ as well, so that adding terms that are identically zero for $[\![u]\!] = 0$ we also have, for every $v \in V_{DG}$ and for every real numbers δ and γ:

$$\widetilde{a}(u, v) - <\{\nabla u\}, [\![v]\!]> - \delta <\{\nabla v\}, [\![u]\!]> + \gamma <[\![u]\!], [\![v]\!]>= (f, v). \tag{5.6}$$

In what follows (as usual for DG methods) we will actually consider only the values $\delta = 1$, $\delta = -1$ and $\delta = 0$, while γ will be assumed to be positive, and represents the usual penalty parameter.

6. Discontinuous VEM. All this will help us in constructing the discrete problem. To start with, for every $K \in \mathcal{T}_h$ we consider again the operator Π_k^K defined in (3.2), and assume that S^K is a bilinear form satisfying the stability property (3.4), that we recall here to avoid confusion with the new notation:

$$c_*(\nabla v, \nabla v)_{0,K} \le S^K(v,v) \le c^*(\nabla v, \nabla v)_{0,K} \quad \forall v \in V_k^K \quad \text{with } \Pi_k^K v = 0 \tag{6.1}$$

We set, for $v,\, w \in H^{s*}(\mathcal{T}_h)$,

$$\begin{aligned} \widetilde{a}_h(v,w) &:= \sum_{K\in\mathcal{T}_h} \widetilde{a}_h^K(v,w) \\ \widetilde{a}_h^K(v,w) &= (\nabla\Pi_k^K v, \nabla\Pi_k^K w)_{0,K} + S^K(v-\Pi_k^K v, w-\Pi_k^K w). \end{aligned} \tag{6.2}$$

THEOREM 6.1. *The bilinear form* (6.2) *satisfies the consistency property* (2.6) *and the stability property* (2.7).

Proof. The proof is exactly the same of Theorem 3.1, and gives (in the new notation)

$$\widetilde{a}_h^K(p,v) = a^K(p,v) \quad \forall p \in \mathbb{P}_k(K),\ \forall v \in (V_{DG})_{|K}, \tag{6.3}$$

$$\alpha_* a^K(v,v) \le \widetilde{a}_h^K(v,v) \le \alpha^* a^K(v,v)\ \forall v \in (V_{DG})_{|K}. \tag{6.4}$$

□

Finally, for $w,\, v \in H^1(\mathcal{T}_h)$ we define the discrete bilinear form as

$$B_h(w,v) := \widetilde{a}_h(w,v) - <\{\nabla\Pi_k w\}, [\![v]\!]> - \delta <\{\nabla\Pi_k v\}, [\![w]\!]> + \gamma <[\![w]\!], [\![v]\!]> . \tag{6.5}$$

In (6.5) δ is, as already said, a parameter to include different DG-schemes. Precisely, for $\delta = 1$ we have the Virtual Element analogue of the $SIPG$ (see [1, 10]), for $\delta = -1$ the analogue of the $NIPG$ (see [8]), and for $\delta = 0$ the analogue of the $IIPG$ ([7, 9]). On the other hand, as we already said, γ is a stabilization parameter that will be assumed to be *big enough*, as usual for DG methods. We also point out that, with an abuse of notation, in (6.5) we denoted by Π_k the operator which, on each element K, coincides with Π_k^K.

THEOREM 6.2. *There exist positive constants M_s and C, independent of h, such that:*

$$B_h(v,v) \ge M_s \|v\|^2_{1,DG} \qquad v \in V_{DG}, \tag{6.6}$$

$$\widetilde{a}_h(v,w) + <[\![v]\!], [\![w]\!]> \le C\|v\|_{1,DG}\|w\|_{1,DG} \quad v,w \in V_{DG}, \tag{6.7}$$

$$<\{\nabla v\}, [\![w]\!]> \le C\|v\|_{2,DG}\|w\|_{1,DG} \quad v \in H^2(\mathcal{T}_h), w \in H^1(\mathcal{T}_h). \tag{6.8}$$

Proof. Following the typical analysis of DG methods (see also [5] in this book) we recall that, using the trace inequality

$$\|v\|^2_{0,\partial K} \leq C\Big(\ell^{-1}\|v\|^2_{0,K} + \ell|v|^2_{1,K}\Big),$$

(ℓ being a characteristic length of K, for instance its diameter) we immediately deduce (6.8). We also notice that, if v is a piecewise polynomial, then (6.8) becomes

$$<\{\nabla v\}, [\![\, w \,]\!]> \leq C\|v\|_{1,DG}\|w\|_{1,DG}, \quad v \text{ p.w. polynomial}, \ w \in H^1(\mathcal{T}_h), \tag{6.9}$$

thanks to (5.4). Inequality (6.7) is an immediate consequence of (6.4). Finally, from (6.4), (6.9) and Cauchy-Scharwz inequality we deduce (6.6) for γ big enough. □

We are now ready to define the discrete problem as follows.

$$\begin{cases} Find\ u_h \in V_{DG}\ such\ that \\ B_h(u_h, v_h) = < f_h, v_h > \ \ \forall v_h \in V_{DG}. \end{cases} \tag{6.10}$$

7. Convergence of DG-VEM. We have the following convergence result.

THEOREM 7.1. *Under Assumptions* **H2***, for γ big enough and for $\delta = 0, 1, -1$ the discrete problem* (6.10) *has a unique solution u_h. Moreover, for every approximation u_I of u in V_{DG} and for every approximation u_π of u that is piecewise in $\mathbb{P}_k$, we have*

$$\|u - u_h\|_{1,DG} \leq C\Big(\|u - u_I\|_{1,DG} + \|u - u_\pi\|_{2,DG} + \|f - f_h\|_{V'_{1,DG}}\Big) \tag{7.1}$$

where C is a constant independent of h.

Proof. Stability (6.6) implies that problem (6.10) has a unique solution $u_h \in V_{DG}$, and

$$\|u_h\|_{1,DG} \leq \frac{\|f\|_0}{M_s}.$$

In order to prove (7.1), set $\eta_h := u_h - u_I$. From (6.6), and then using (6.10) and (6.5), we have:

$$\begin{aligned} M_s\,\|\eta_h\|^2_{1,DG} \leq B_h(\eta_h, \eta_h) &= B_h(u_h, \eta_h) - B_h(u_I, \eta_h) \\ &= \Big(< f_h, \eta_h > - \widetilde{a}_h(u_I, \eta_h) + < \{\nabla \Pi^k u_I\}, [\![\, \eta_h \,]\!] > \Big) \\ &\quad + \Big(\delta < \{\nabla \Pi^k \eta_h\}, [\![\, u_I \,]\!] > - \gamma < [\![\, u_I \,]\!], [\![\, \eta_h \,]\!] \Big) =: I + II. \end{aligned} \tag{7.2}$$

Adding and subtracting u_π, and then using (6.3) we have:

$$\begin{aligned} I =& < f_h, \eta_h > - \sum_K \Big(\widetilde{a}_h^K(u_I - u_\pi, \eta_h) + \widetilde{a}_h^K(u_\pi, \eta_h) \Big) \\ &+ <\{\nabla \Pi_k^K (u_I - u_\pi)\}, [\![\eta_h]\!]> + < \{\nabla \Pi_k^K u_\pi\}, [\![\eta_h]\!] > \\ =& < f_h, \eta_h > - \sum_K \Big(\widetilde{a}_h^K(u_I - u_\pi, \eta_h) + a^K(u_\pi, \eta_h) \Big) \\ &+ <\{\nabla \Pi_k^K (u_I - u_\pi)\}, [\![\eta_h]\!]> + < \{\nabla \Pi_k^K u_\pi\}, [\![\eta_h]\!] > . \end{aligned} \tag{7.3}$$

Then we add the term $\widetilde{a}(u, \eta_h) - < \{\nabla u\}, [\![\eta_h]\!] > -(f, \eta_h)$ that, thanks to (5.5)), is equal to zero, and in the last term we remember that, thanks to (3.3), $\Pi_k^K u_\pi = u_\pi$. We obtain

$$\begin{aligned} I =& < f_h, \eta_h > - \sum_K \Big(\widetilde{a}_h^K(u_I - u_\pi, \eta_h) + a^K(u_\pi, \eta_h) \Big) \\ &+ \widetilde{a}(u, \eta_h) - < \{\nabla u\}, [\![\eta_h]\!] > -(f, \eta_h) \\ &+ <\{\nabla \Pi_k^K (u_I - u_\pi)\}, [\![\eta_h]\!]> + < \{\nabla u_\pi\} [\![\eta_h]\!] >, \end{aligned} \tag{7.4}$$

that rearranging terms we write as

$$\begin{aligned} I =& < f_h, \eta_h > -(f, \eta_h) - \sum_K \Big(\widetilde{a}_h^K(u_I - u_\pi, \eta_h) + a^K(u_\pi - u, \eta_h) \Big) \\ &+ < \{\nabla \Pi_k^K (u_I - u_\pi)\}, [\![\eta_h]\!] > + < \{\nabla (u_\pi - u)\}, [\![\eta_h]\!] > . \end{aligned} \tag{7.5}$$

Using (6.9) and (6.8) in (7.5) we have then

$$|I| \le C \Big(\|f - f_h\|_{V'_{1,DG}} + \|u_I - u_\pi\|_{1,DG} + \|u_\pi - u\|_{2,DG} \Big) \|\eta_h\|_{1,DG}. \tag{7.6}$$

On the other hand, recalling first that $[\![u]\!] = 0$, and then using (5.4) we have

$$\begin{aligned} |II| &= \Big| \delta < \{\nabla \Pi^k \eta_h\}, [\![u_I - u]\!] > -\gamma < [\![u_I - u]\!], [\![\eta_h]\!] \Big| \\ &\le C \, \|u - u_I\|_{1,DG} \|\eta_h\|_{1,DG}. \end{aligned} \tag{7.7}$$

Using (7.6) and (7.7) in (7.2) we have then

$$\|\eta_h\|_{1,DG} \le C \Big(\|f - f_h\|_{V'_{1,DG}} + \|u - u_I\|_{1,DG} + \|u_\pi - u\|_{2,DG} \Big), \tag{7.8}$$

and estimate (7.1) follows by triangle inequality.
□

According to the classical Scott-Dupont theory (see e.g. [4]) we have the following result.

PROPOSITION 7.1. *Assume that Assumption* **H1** *is satisfied. Then there exists a constant* C*, depending only on* k *and* ζ*, such that for every* $w \in H^{k+1}(K)$ *there exist a* $w_\pi \in \mathbb{P}_k(K)$*, and a* $w_I \in V_k^K$ *such that*

$$\begin{aligned} |w - w_\pi|_{r,K} &\le C\, h_K^{k+1-r}\, |w|_{k+1,K} \quad 0 \le r \le k+1, \\ |w - w_I|_{r,K} &\le C\, h_K^{k+1-r}\, |w|_{k+1,K} \quad r = 0, 1. \end{aligned} \tag{7.9}$$

This, together with (4.2), inserted in (7.1) gives the optimal estimate

$$\|u - u_h\|_{1,DG} \le C\, h^k |u|_{k+1,\Omega}.$$

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