Applications of the Pseudo Residual-Free Bubbles to the Stabilization of Convection-Diffusion Problems

F. Brezzi[1,2], *D. Marini*[1,2], *and A. Russo*[2]

Abstract

Residual-free bubbles have been recently introduced in order to compute optimal values for the stabilization methods à la Hughes-Franca. However, unless in very special situations, (one-dimensional problems, limit cases, etc.) they require the actual solution of PDE problems (the bubble problems) in each element. Thus they are very difficult to be used in practice. In this paper we present, for the special case of convection-dominated elliptic problems, a cheap way to compute approximately the solution of the bubble problem in each element. This provides, as a consequence, a cheap way to compute good approximations for the optimal values of the stabilization parameters.

1 Introduction

We will present in this paper a new stabilization method for convection-diffusion problems, particularly designed to treat strongly convection-dominated problems, but able to adapt naturally from diffusion-dominated regime to convection-dominated regime in a very simple way. We will consider, for the sake of simplicity, the following linear elliptic convection-diffusion problem in a polygonal domain Ω:

$$\begin{cases} \mathcal{L}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where

$$\mathcal{L}u = -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u. \tag{2}$$

Let $\mathcal{T}_h = \{K\}$ be a family of regular discretizations of Ω into triangles K, and let $h_K = \mathrm{diam}(K)$, $h = \max_K h_K$. We assume that the diffusion ε is a positive constant, and both the convection field $\boldsymbol{a}$ and the right-hand side f are piecewise constant with respect to the triangulation $\mathcal{T}_h$. If the operator $\mathcal{L}$ is *convection-dominated*, it is well known that the exact solution of (1) can exhibit boundary and internal *layers*, i.e., very narrow regions where the solution and its derivatives change abruptly. As a consequence,

[1]Dipartimento di Matematica, Università di Pavia, Italy
[2]Istituto di Analisi Numerica del CNR, Pavia, Italy

if we employ a classical finite element method with a discretization scale which is too big to resolve the layers, the solution that we get has in general large numerical oscillations spreading all over the domain, and can be completely unrelated to the true solution. To properly resolve the layers, the discretization parameter must be of the same size of the ratio between diffusion and convection. In many problems, this choice would lead to a huge number of degrees of freedom, making the discretization intractable. In recent years, many stabilization methods have been invented to cope with this kind of problem. Among them, let us recall the SUPG method (Streamline-Upwind Petrov/Galerkin), first described in [5], which has been successfully applied to many different situations (see e.g. [6] and the references therein). As it is well known, the method corresponds to adding a consistent term providing an additional diffusion in the streamline direction (see (10) below). The amount of such additional diffusion is tuned by a parameter τ that must be chosen in a suitable way. According to thumb-rule arguments and a lot of numerical tests, several recipes have been proposed for the choice of τ (one of them being recalled in (11) and (12) below). The method has been proven to have a solid mathematical basis in several cases of practical interest (see e.g. [9]). Nevertheless, the need for a suitable convincing argument to guide the choice of τ is still considered as a major drawback of the method by several users. In recent times, SUPG has been related to the process of addition and elimination of suitable *bubble functions* (see [2, 1]) that aroused considarable interest, although the problem of the optimal choice of τ was simply translated into the problem of the optimal choice of the bubble space. In more recent times, however, starting from [4] and further developed in [7], a guideline for the choice of optimal bubble spaces came into the market. Roughly speaking this approach, called *residual-free bubbles* method, related the shape of the "optimal bubble space" (the one giving rise to the "optimal τ") to the solution of suitable boundary value problems (obviously, strictly related to the original one in Ω) in each element K. In our case this would correspond to solving, in each K, the following boundary-value problem:

$$\begin{cases} \text{find } b_K \in H^1_0(K) \text{ such that} \\ \mathcal{L} b_K = 1 \text{ in } K \end{cases} \tag{3}$$

(which is, in a sense, as difficult as (1)), and then setting

$$\tau = \frac{1}{|K|} \int_K b_K. \tag{4}$$

It was surely reassuring to see that one arrives to the same conclusion starting from a different point of view, basically aiming at taking into account the effect of unresolvable scales onto the resolvable ones (see [8] and [3]), but both approaches require at the end the solution of the same boundary value problem in each K. For the limit case $\varepsilon \to 0$, one can compute the limit solution in some special cases (included the present one), but a general approach is still lacking.

We present here a methodology for computing, at a very cheap price, an approximate solution of (3), in the hope that it is "good enough" to indicate suitable values for the parameter τ. As we shall see in the following with more details, our approximate optimal bubble (that we call *pseudo residual-free bubble*) is set a priori to be piecewise linear, inside K, onto the simple mesh generated by taking a point P internal to K and connecting it to the vertices of K (in this way we obtain three subtriangles). A reasonable cheap algorithm dictates a suitable choice for the location of P. The method that we generate in this way increases smoothly the amount of added streamline diffusion as ε decreases, and gives the same limit (for $\varepsilon \to 0$) as the residual-free bubbles.

The layout of the paper is as follows. In Section 2 we briefly recall the basic ideas of the SUPG method. In Section 3 we describe the residual-free bubbles method which is the starting point for deriving the pseudo-residual-free bubbles method, described in Section 4.

2 The SUPG Method

In this Section we will briefly present the SUPG stabilization method for our convection-diffusion problem (1) (2) (see [6]). For, let us recall the classical variational formulation of problem (1):

$$\begin{cases} \text{find } u \in H^1_0(\Omega) \text{ such that} \\ a\,(u,v) = F(v) \text{ for all } v \in H^1_0(\Omega) \end{cases} \tag{5}$$

where

$$a\,(u,v) = \varepsilon \int_\Omega \nabla u \cdot \nabla v + \int_\Omega (\boldsymbol{a} \cdot \nabla u)\, v \tag{6}$$

is a continuous and coercive bilinear form on the Hilbert space $H^1_0(\Omega)$ and

$$v \mapsto F(v) = \int_\Omega f v \tag{7}$$

is in $H^{-1}(\Omega)$. A Galerkin approximation of problem (1) consists in taking a finite-dimensional subspace V_h of $H^1_0(\Omega)$, and then solving the variational problem (5) in V_h. For the sake of simplicity, from now on we will restrict ourselves to the case of *continuous, piecewise linear* elements, i.e., we will consider the finite element space

$$V_L = \left\{ v \in H^1_0(\Omega),\ v_{|K} \text{ linear for all } K \in \mathcal{T}_h \right\} \tag{8}$$

so that the approximation of (5) reads

$$\begin{cases} \text{find } u_L \in V_L \text{ such that} \\ a\,(u_L, v_L) = F(v_L) \text{ for all } v_L \in V_L. \end{cases} \tag{9}$$

As already pointed out, if the problem is convection-dominated, then, unless h is of the same size of ε, the solution of (9) will exhibit strong oscillations spreading all over the domain. The SUPG method consists in adding to the original bilinear form $a\,(\cdot,\cdot)$ a term which introduces a suitable amount of artificial diffusion in the direction of streamlines, but without upsetting consistency. In the case of problem (1) (and with linear elements) the SUPG method reads

$$\begin{cases} \text{find } u_L \in V_L \text{ such that for all } v_L \in V_L \\ a\left(u_L, v_L\right) + \sum_K \tau_K \int_K \left(\boldsymbol{a} \cdot \nabla u_L - f\right)\left(\boldsymbol{a} \cdot \nabla v_L\right) = F(v_L), \end{cases} \tag{10}$$

where τ_K is a stabilization parameter depending on the local character of the discretization: in elements whose diameter is not small enough to resolve all scales, $\tau_K \approx h_K/|\boldsymbol{a}_{|K}|$ and elsewhere $\tau_K \approx 0$. More precisely, we can introduce a mesh Péclet number in the following way:

$$\text{for each } K \in \mathcal{T}_h, \quad \mathrm{Pe}_K = \frac{|\boldsymbol{a}_{|K}| h_K}{6\varepsilon}, \tag{11}$$

and then define τ_K element by element accordingly to the size of Pe_K:

$$\tau_K = \begin{cases} \dfrac{h_K}{2|\boldsymbol{a}_{|K}|} & \text{if } \mathrm{Pe}_K \geq 1 \\ \dfrac{h_K^2}{12\varepsilon} & \text{if } \mathrm{Pe}_K < 1. \end{cases} \tag{12}$$

Scheme (10) leads to a reasonable numerical solution, where of course layers are not resolved, but they are very well localized, and away from the layers the accuracy is very good. We refer to [6, 9] for further details.

A possible drawback of the SUPG method is the sensitivity of the solution to the stabilization parameter τ_K, whose value is not determined precisely by the available theory. A way to recover intrinsically the value of τ_K is to use the residual-free bubbles approach (see [7, 3]) that will be recalled in the next Section.
The effect of stabilization obtained by modifying the original bilinear form as in (10) can be obtained by enlarging the finite element space in the following way.

For each element K, we define the space of bubbles in K as

$$B_K = H_0^1(K) \tag{13}$$

and the enlarging space V_B as

$$V_B = \oplus_K B_K. \tag{14}$$

Then, we solve problem (5) on

$$V_h = V_L \oplus V_B. \tag{15}$$

Of course, we cannot pretend to solve *exactly* problem (5) in V_h, because V_h is infinite-dimensional. We will make some approximations later on.

By (15) we have that any $v_h \in V_h$ can be split into a linear part $v_L \in V_L$ and into a bubble part $v_B \in V_B$ in a unique way:

$$v_h = v_L + v_B \in V_L \oplus V_B, \tag{16}$$

and the bubble part itself can be uniquely split element by element:

$$v_B = \sum_K v_{B,K}, \qquad v_{B,K} \in B_K. \tag{17}$$

Then, the variational problem (5) in V_h can be written as follows:

$$\begin{cases} \text{find } u_h = u_L + u_B \in V_L \oplus V_B \text{ such that} \\ \text{for all } v_L \in V_L,\ K \in \mathcal{T}_h, \text{ and } v_{B,K} \in B_K \\ a\left(u_L + u_B, v_L\right) = F(v_L) \\ a\left(u_L + u_{B,K}, v_{B,K}\right)_K = F(v_{B,K})_K, \end{cases} \tag{18}$$

where the subscript $(\cdot)_K$ indicates that the integrals involved are restricted to the element K. Consider the first equation of (18); using the decomposition (17) on u_B and the bilinearity of $a\left(\cdot,\cdot\right)$, it can be written as

$$a\left(u_L, v_L\right) + \sum_K a\left(u_{B,K}, v_L\right)_K = F(v_L). \tag{19}$$

The term $\sum_K a\left(u_{B,K}, v_L\right)_K$ represents the effect of the bubble part $u_{B,K}$ onto the linear part u_L. We can give this term a different expression, observing that

$$a\left(u, v\right)_K = \left(\mathcal{L}u, v\right)_K = \left(u, \mathcal{L}_K^* v\right)_K \tag{20}$$

where $\mathcal{L}_K^*$ is the formal adjoint of $\mathcal{L}$ on K (with zero boundary conditions on ∂K). We then have

$$\sum_K a\left(u_{B,K}, v_L\right)_K = \sum_K \left(u_{B,K}, \mathcal{L}_K^* v_L\right)_K. \tag{21}$$

We now use the second equation in (18) to determine $u_{B,K}$ in terms of u_L. By linearity, we can rewrite it as

$$a\left(u_{B,K}, v_{B,K}\right)_K = -\left[a\left(u_L, \cdot\right) - F(\cdot)\right]_K \left(v_{B,K}\right) \tag{22}$$

or, using the differential operator (recall that the test functions $v_{B,K}$ range on the whole space $H_0^1(K)$),

$$\begin{cases} \mathcal{L}u_{B,K} = -[\mathcal{L}u_L - f] & \text{in } K \\ \qquad u = 0 & \text{on } \partial K. \end{cases} \tag{23}$$

For each u_L, problem (22) (or (23)) has always a unique solution $u_{B,K} \in B_K$ which can be written as

$$u_{B,K} = M_K\left(\mathcal{L}u_L - f\right), \tag{24}$$

where M_K is a bounded linear operator from $H^{-1}(K)$ to $B_K = H_0^1(K)$. Substituting (24) into (19) the equation for u_L becomes:

$$a\left(u_L, v_L\right) + \sum_K a\left(M_K\left(\mathcal{L}u_L - f\right), v_L\right)_K = \left(f, v_L\right), \tag{25}$$

or, using (20):

$$a\left(u_L, v_L\right)+\underbrace{\sum_K\left(M_K\left(\mathcal{L} u_L-f\right), \mathcal{L}_K^* v_L\right)_K}_{\text{effect of residual-free bubbles onto linears}}=\left(f, v_L\right) \tag{26}$$

for all $v_L \in V_L$. Since the coefficients of the operator are piecewise constant, for each $K \in \mathcal{T}_h$ we have

$$\mathcal{L}_K^* u=-\varepsilon \Delta u-\boldsymbol{a}_{|K} \cdot \nabla u. \tag{27}$$

Recall that V_L is the space of continuous, piecewise linear elements on $\mathcal{T}_h$. Then

$$\left(\mathcal{L} u_L-f\right)_{|K}=\left(\boldsymbol{a} \cdot \nabla u_L-f\right)_{|K}=\text{constant} \tag{28}$$

and

$$\left(\mathcal{L}_K^* v_L\right)_{|K}=-\left(\boldsymbol{a} \cdot \nabla v_L\right)_{|K}=\text{constant}. \tag{29}$$

As a consequence, we have

$$\begin{aligned}
&\left(M_K\left(\mathcal{L} u_L-f\right), \mathcal{L}_K^* v_L\right)_K= \\
&-\left(\boldsymbol{a} \cdot \nabla u_L-f\right)_{|K}\left(\boldsymbol{a} \cdot \nabla v_L\right)_{|K}\left(M_K(1), 1\right)_K= \\
&\left(\boldsymbol{a} \cdot \nabla u_L-f\right)_{|K}\left(\boldsymbol{a} \cdot \nabla v_L\right)_{|K} \int_K M_K(-1)= \\
&\left[\int_K\left(\boldsymbol{a} \cdot \nabla u_L-f\right)\left(\boldsymbol{a} \cdot \nabla v_L\right)\right] \frac{1}{|K|} \int_K M_K(-1).
\end{aligned} \tag{30}$$

The resulting scheme on V_L is then

$$\left\{\begin{array}{l}
\text{find } u_L \in V_L \text{ such that for all } v_L \in V_L \\
a\left(u_L, v_L\right)+\sum_K \widehat{\tau}_K \int_K\left(\boldsymbol{a} \cdot \nabla u_L-f\right)\left(\boldsymbol{a} \cdot \nabla v_L\right)=F\left(v_L\right)
\end{array}\right. \tag{31}$$

where

$$\widehat{\tau}_K=\frac{1}{|K|} \int_K M_K(-1). \tag{32}$$

We see that the SUPG scheme (10) and (31) have an identical structure; we need only to compare the two constants τ_K and $\widehat{\tau}_K$. By (24) we see that

$$M_K(-1)=b_K, \tag{33}$$

where b_K solves the following boundary value problem on K:

$$\left\{\begin{aligned}
\mathcal{L} b_K &=1 \quad \text{in } K \\
b_K &=0 \quad \text{on } \partial K,
\end{aligned}\right. \tag{34}$$

i.e.

$$\left\{\begin{aligned}
-\varepsilon \Delta b_K+\boldsymbol{a}_{|K} \cdot \nabla b_K &=1 \quad \text{in } K \\
b_K &=0 \quad \text{on } \partial K.
\end{aligned}\right. \tag{35}$$

We are left with the problem of evaluating, possibly in some approximate way, the integral of b_K appearing in (32). For strongly convection-dominated cases (the most interesting ones) we can argue as in [4]. If $\varepsilon \ll |\boldsymbol{a}_{|K}|\, h_K$, the solution of (35) will be very close to a pyramid with one (or two) almost vertical faces on the outflow boundary of K (the element boundary layer). The remaining faces of this pyramid have slope $1/|\boldsymbol{a}_{|K}|$ in the direction of $\boldsymbol{a}_{|K}$. Hence, if we define $\widehat{h}_K$ as the length of longest segment parallel to $\boldsymbol{a}_{|K}$ and contained in K, we have

$$\int_K b_K \approx \text{Volume of the pyramid} = \frac{|K|}{3}\,\frac{\widehat{h}_K}{|\boldsymbol{a}_{|K}|}, \tag{36}$$

so that

$$\widehat{\tau}_K = \frac{1}{|K|}\int_K b_K \approx \frac{\widehat{h}_K}{3\,|\boldsymbol{a}_{|K}|}. \tag{37}$$

Using a scaling argument (see [10]), we can also show that when ε is large with respect to $|\boldsymbol{a}_{|K}| h_K$, we have

$$\frac{1}{|K|}\int_K b_K \approx C\frac{h_K^2}{\varepsilon} \tag{38}$$

where C still depends on K and h but can be uniformly bounded from above and from below if we have a regular family of triangulations.

If we compare (12) and (37) we see that the values of τ_K and $\widehat{\tau}_K$ are very close in both limits; indeed, the theoretical results for the SUPG method of [6, 9] also hold with $\widehat{\tau}_K$ in place of τ_K, and the numerical experiments give results of similar quality.

3 The Pseudo Residual-Free Bubbles

As we have seen in the previous section, the problem of finding the *optimum* value for τ_K would be solved if we knew explicitly, in each triangle K and for any given value of ε and $\boldsymbol{a}_{|K}$, the exact solution of problem (35) (or, at least, its integral on K). However, in general, this cannot be computable in an easy way. In this section we shall present a strategy to solve this problem, at least in a reasonably good approximate way. The idea is to look for a solution of (35) having the shape of a pyramid, with vertex in a point P internal to K, to be chosen in order to minimize the L^1-norm of the residual. More precisely, let P be (any) internal point of K, let V_i $(i = 1, 2, 3)$ be the vertices of K ordered, as usual, counterclockwise, and let $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ be the edges of K, with $\boldsymbol{e}_i$ opposite to V_i (see Fig. 1). We denote by $K_1,\ K_2,\ K_3$ the three subtriangles obtained by connecting P with the vertices V_i. Consider a function vanishing on ∂K, continuous on $\overline{K}$, and piecewise linear on each K_i. Clearly, $\forall P \in K$ there is only one function of this type with value 1 in P. Let $b_P(x)$ be such a function. We want now to approximate the solution b_K of (35) with functions of the type $\alpha b_P(x)$, where α and P have to be suitably chosen. As a first step, we choose α as a function of P; we look for $\alpha = \alpha(P)$ such that

$$a\,(\alpha b_P, b_P)_K = \int_K b_P. \tag{39}$$

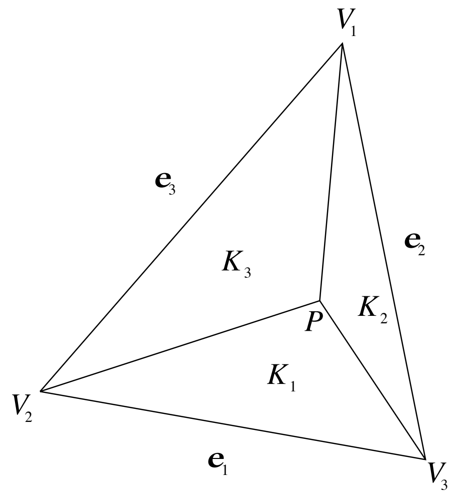

FIGURE 1
Notation

Note that this is a Galerkin approximation of (35) in the 1-dimensional space spanned by b_P. Since $\boldsymbol{a}_{|K}$ is constant, an easy computation gives

$$\alpha(P) = \frac{\int_K b_P}{\varepsilon \int_K |\nabla b_P|^2}. \tag{40}$$

We then set $B_P(x) = \alpha(P) b_P(x)$. (Notice that $\alpha(P)$ does not depend on the convection coefficient.) As a second step, we want to choose P. For, we require that

$$\int_K |-\varepsilon \Delta B_P + \boldsymbol{a}_{|K} \cdot \nabla B_P - 1| \tag{41}$$

is minimum. Note that, B_P being piecewise linear on K, the term ΔB_P will have only a distributional meaning, so that the integral appearing in (41) has to be intended in the sense of measures.

As we shall see, there are infinitely many points P minimizing (41). The choice of the "best" P among them will be discussed later on.

Let us now turn to the computational aspects. We denote by e_i, $(i = 1, 2, 3)$, the length of the edge $\boldsymbol{e}_i$ and by λ_i $(i = 1, 2, 3)$ the (unknown) barycentric coordinates of P, given by

$$\lambda_i = \lambda_i(P) = |K_i|/|K|.$$

With this notation, and since $\nabla b_{P|K_i} = -e_i \underline{n}^i / 2|K_i|$, (where $\underline{n}^i$ is the outward normal to $\boldsymbol{e}_i$), standard computations give

$$\begin{aligned} \int_K b_P(x) &= |K|/3, \\ \int_K |\nabla b_P|^2 &= \sum_j \int_{K_j} |\nabla b_P|^2 = \\ \sum_j \int_{K_j} \frac{e_j^2}{4|K_j|^2} &= \sum_j (e_j^2/4|K_j|), \end{aligned}$$

so that, from (40)

$$\alpha(P) = \frac{|K|/3}{\varepsilon \sum_j (e_j^2/4|K_j|)} = \frac{4|K|^2}{3\varepsilon \sum_j (e_j^2/\lambda_j)}. \tag{42}$$

We now turn to the minimization of (41). We first remark that ΔB_P is a measure concentrated on $(\cup \partial K_i) \setminus \partial K$, while $(\boldsymbol{a}_{|K} \cdot \nabla B_P - 1)$ is an integrable function. This implies that

$$\begin{aligned} &\int_K | -\varepsilon \Delta B_P + \boldsymbol{a}_{|K} \cdot \nabla B_P - 1| = \\ &\int_K | -\varepsilon \Delta B_P| + \int_K |\boldsymbol{a}_{|K} \cdot \nabla B_P - 1| \end{aligned} \tag{43}$$

where only the last integral is in the ordinary sense (the others being in the sense of measures.) Moreover, a little functional analysis plus usual computations lead to

$$\begin{aligned} &\int_K | -\varepsilon \Delta B_P| = -\varepsilon \int_{\partial K} \frac{\partial B_P}{\partial n}\, d\Gamma = \\ &\varepsilon \alpha(P) \sum_i \frac{\boldsymbol{e}_i^2}{2|K_i|} = \frac{2}{3}\,|K| \end{aligned} \tag{44}$$

so that minimizing (41) amounts to minimize

$$\begin{aligned} J(P) \ &= \int_K |\boldsymbol{a}_{|K} \cdot \nabla B_P - 1| = \\ \sum_i \int_{K_i} &|\boldsymbol{a}_{|K} \cdot \nabla B_P - 1| =: \sum_i \int_{K_i} |g_i|, \end{aligned} \tag{45}$$

where, for every fixed P, $g_i = g_i(P)$ are constants defined as

$$g_i := (\boldsymbol{a}_{|K} \cdot \nabla B_P - 1)_{|K_i} \qquad (i = 1, 2, 3). \tag{46}$$

We notice now that

$$\sum_i \int_{K_i} \boldsymbol{a}_{|K} \cdot \nabla B_P = 0 \tag{47}$$

so that, for all $P \in K$

$$\sum_i \int_{K_i} g_i = \int_K -1 \equiv -|K| \tag{48}$$

and finally, always for every $P \in K$,

$$\sum_i \int_{K_i} |g_i| \geq |K|. \tag{49}$$

At least one of the g_i's is ≤ 0 (from (48)). If, for some $\tilde{P} \in K$, all the g_i's are ≤ 0, then $\tilde{P}$ minimizes (41), thanks to (48) and (49). Next, we show that there is always a subset S of K (with positive measure) such that, for every P in S, we have

$$g_i(P) \ \leq \ 0 \qquad (i = 1, 2, 3), \tag{50}$$

and afterwards we choose P in S. To prove (50) we rewrite it as

$$\Big(\alpha(P)\boldsymbol{a}_{|K} \cdot \nabla b_P\Big)_{|K_i} \leq 1, \tag{51}$$

or, using definition (42) for $\alpha(P)$, as

$$4|K|^2\boldsymbol{a}_{|K} \cdot \nabla b_{P|K_i} \leq 3\varepsilon \sum_j (e_j^2/\lambda_j), \tag{52}$$

and finally, since $\nabla b_{P|K_i} = -e_i\underline{n}^i/2|K_i| = -|K|e_i\underline{n}^i/2\lambda_i$,

$$-(\boldsymbol{a}_{|K} \cdot \underline{n}^i)e_i \leq \frac{3\varepsilon}{2|K|}\lambda_i \sum_j (e_j^2/\lambda_j), \tag{53}$$

that we rewrite, with more compact notation, as

$$b_i \leq c\lambda_i(P)\varphi(P), \tag{54}$$

where we have set

$$b_i = -(\boldsymbol{a}_{|K} \cdot \underline{n}^i)e_i, \; c = \frac{3\varepsilon}{2|K|}, \; \varphi(P) = \sum_j (e_j^2/\lambda_j). \tag{55}$$

We point out explicitly that b_i and c are independent of P. We also point out that

$$\sum_i b_i = -\sum_i (\boldsymbol{a}_{|K} \cdot \underline{n}^i)e_i = -\int_{\partial K} \boldsymbol{a}_{|K} \cdot \underline{n}\, d\Gamma = 0, \tag{56}$$

(since $\boldsymbol{a}_{|K}$ is constant), and therefore $b_i \leq 0$ for at least one index i. We also notice that

$$c\lambda_i(P)\varphi(P) \geq 0 \qquad \forall P \in K, \; \forall i \tag{57}$$

so that (54) trivially holds all over K for all the indices i such that $b_i \leq 0$. Finally, we have

$$\varphi(P) \to +\infty \qquad \text{for } P \to \partial K, \tag{58}$$

so that for every $i = 1, 2, 3$ the product $\lambda_i(P)\varphi(P)$ still goes to plus infinity uniformly on any compact subset of ∂K that does not contain $\overline{\boldsymbol{e}}_i$. With all these remarks in mind, consider first the case where only one of the b_i is ≤ 0. To fix ideas, let $b_1 \leq 0$ while $b_2 > 0, \; b_3 > 0$. In this case, we easily see that both $\lambda_2(P)\varphi(P)$ and $\lambda_3(P)\varphi(P)$ go to plus infinity when P tends to any fixed point internal to the edge $\boldsymbol{e}_1$, so that (54) holds for all i's on a set S close to it. Take then the line through V_1 parallel to $\boldsymbol{a}_{|K}$. This line crosses the edge $\boldsymbol{e}_1$ in a point P_1 different from V_2 and V_3, and along this line we have $(\boldsymbol{a}_{|K} \cdot \nabla B_P)_{|K_2} = (\boldsymbol{a}_{|K} \cdot \nabla B_P)_{|K_3}$ for obvious continuity reasons. Hence, we define in this case P^* as the point closest to P_1, on the line V_1P_1, among the points of S. Next, consider the case where two of the b_i are ≤ 0. To fix ideas, let $b_1 \leq 0, \; b_2 \leq 0$, while $b_3 > 0$. In this case (54) holds all over K for $i = 1$ and $i = 2$, and we see that $\lambda_3(P)\varphi(P)$ goes to plus infinity when P tends to the vertex V_3, so that (54) holds also for $i = 3$ in a region

S close to V_3. Let then P_3 be the midpoint of the edge $\boldsymbol{e}_3$ and take the line V_3P_3. In this case we define P^* as the point closest to V_3, on the line V_3P_3, among the points of S. It is easy to see that, in practice, the actual computation of P^* is extremely simple, and can be performed at a very cheap cost. Similarly, the computation of the $\widehat{\tau}_K$ corresponding to the *pseudo residual-free bubble* B_{P^*}, denoted by $\widehat{\tau}_K^*$, is also quite economic, and is again given by

$$\widehat{\tau}_K^* = \frac{1}{|K|}\int_K B_{P^*}. \tag{59}$$

We remark explicitly that, with the use of the stabilization coefficient $\widehat{\tau}_K^*$ given by the pseudo residual-free bubble, we are adding a streamline diffusion term which grows smoothly and in a very natural way as $\varepsilon/|\boldsymbol{a}_{|K}|h$ approaches zero, i.e., as the problem becomes convection-dominated. We also point out that, for fixed K and $\boldsymbol{a}_{|K}$, denoting by

$$\widehat{\tau}_K(\varepsilon) = \frac{1}{|K|}\int_K b_K \tag{60}$$

and

$$\widehat{\tau}_K^*(\varepsilon) = \frac{1}{|K|}\int_K B_{P^*} \tag{61}$$

the coefficients provided by the residual-free bubble and by the pseudo residual-free bubble respectively, we easily have

$$\lim_{\varepsilon\to 0}\widehat{\tau}_K(\varepsilon) = \lim_{\varepsilon\to 0}\widehat{\tau}_K^*(\varepsilon), \tag{62}$$

both limits being equal to the volume of the pyramid divided by $|K|$ as discussed at the end of the previous Section.

4 Numerical Experiments

In the Section we will present a numerical experiment showing the effect of the stabilization coefficient computed with the pseudo residual-free bubble. As we have noticed before, the pseudo residual-free bubble scheme coincides with the residual-free bubble scheme in the convection-dominated limit $\varepsilon \to 0$, so that a comparison with other methods should be made in the intermediate regime. We will compare the pseudo residual-free bubble method with the classical SUPG method described in Section 2.

We will solve the following problem:

$$\begin{cases} -\varepsilon\Delta u + \boldsymbol{a}\cdot\nabla u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega, \end{cases} \tag{63}$$

where the computational domain Ω and the Dirichlet boundary data g are as in Fig. 2, $\varepsilon = 5\cdot 10^{-3}$ and $\boldsymbol{a}(x,y) = (-y,x)$ is shown in Fig. 3. We will use the mesh shown in Fig. 4, for which the local Peclet number defined in (11) is partly > 1 and partly < 1, as shown in Fig. 5. Hence the problem is partly diffusion-dominated and partly convection-dominated. The level curves of the solution obtained with the SUPG stabilization and with the pseudo residual-free bubbles stabilization are shown in Fig. 6 and Fig. 7 respectively. In Fig. 8 is shown the solution to the same problem obtained with the plain Galerkin method (no stabilization) but with a mesh fine enough to resolve all the fine details of the solution. The mesh used has 11639 elements and is reported in Fig. 9. We notice that the SUPG solution presents a slight over-diffusivity in the reentrant corner region, white the pseudo residual-free bubbles solution is qualitatively closer to the "exact solution" of Fig. 8. At a first sight, this difference could seem quite small, but we have to keep in mind that the solution of a linear scalar convection-diffusion equation is only one small step of the long way to solving the full Navier-Stokes equations. Hence, small differences could become important in a more complex situation. We think that this point should be analyzed in much more details.

5 Conclusions

The residual-free bubbles technique interprets the stabilization parameter as the mean value of the solution of a differential equation defined at the element level. The value of the parameter can be easily determined in the convection-dominated limit; for intermediate cases, a simple minded interpolation is usually employed. This paper is a first attempt toward the computation of the parameter in the intermediate regime. We have introduced the concept of "pseudo residual-free bubble", which is a certain approximation of the true residual-free bubble. We have shown that in some cases the method obtained by using the pseudo residual-free bubbles gives better results than the SUPG method.

References

[1] C. Baiocchi, F. Brezzi, and L. Franca, *Virtual bubbles and the Galerkin-least-squares method*, Comput. Methods Appl. Mech. Engrg., 105 (1993), pp. 125–141.

[2] F. Brezzi, M. Bristeau, L. Franca, M. Mallet, and G. Rogé, *A relationship between stabilized finite element methods and the Galerkin method with bubble functions*, Comput. Methods Appl. Mech. Engrg., 96 (1992), pp. 117–129.

[3] F. Brezzi, L. P. Franca, T. J. R. Hughes, and A. Russo, $b = \int g$, Comput. Methods Appl. Mech. Engrg., 142 (1997), pp. 353–360.

[4] F. Brezzi and A. Russo, *Choosing bubbles for advection-diffusion problems*, Math. Models Meth. Appl. Sci., 4 (1994), pp. 571–587.

[5] A. N. Brooks and T. J. R. Hughes, *Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 32 (1982), pp. 199–259.

[6] L. P. Franca, S. L. Frey, and T. J. R. Hughes, *Stabilized finite element methods: I. Application to the advective-diffusive model*, Comput. Methods Appl. Mech. Engrg., 95 (1992), pp. 253–276.

[7] L. P. Franca and A. Russo, *Deriving upwinding, mass lumping and selective reduced integration by residual-free bubbles*, Appl. Math. Letters, 9 (1996), pp. 83–88.

[8] T. J. R. Hughes, *Multiscale phenomena: Green's functions, the Dirichlet-to-Neumann formulation, subgrid scale models, bubbles and the origin of stabilized methods*, Comput. Methods Appl. Mech. Engrg., 127 (1995), pp. 387–401.

[9] C. Johnson, U. Nävert, and J. Pitkäranta, *Finite element methods for linear hyperbolic problem*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.

[10] A. Russo, *Bubble stabilization of finite element methods for the linearized incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg., 132 (1996), pp. 335–343.

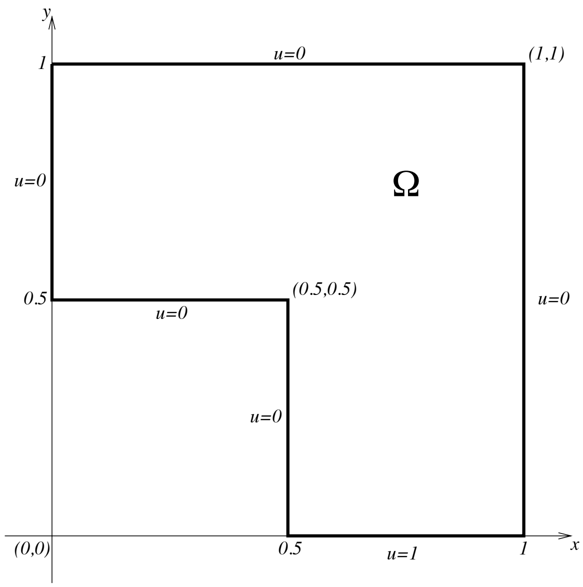

FIGURE 2
Domain and boundary data for the test problem

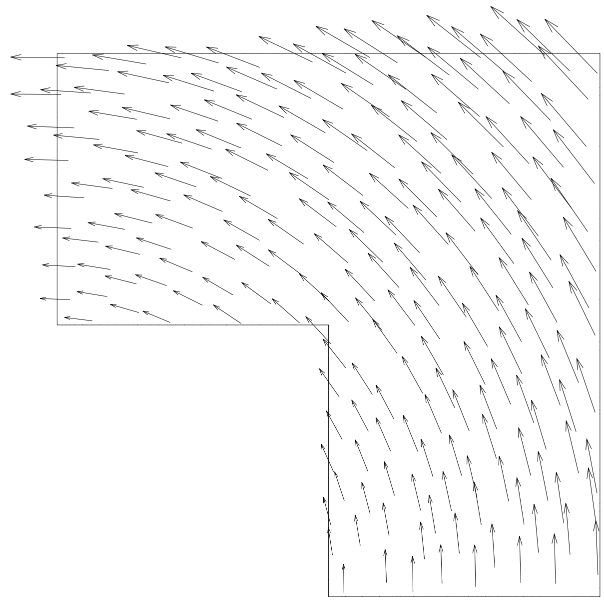

FIGURE 3
The convection field $\boldsymbol{a}$ *for the test problem*

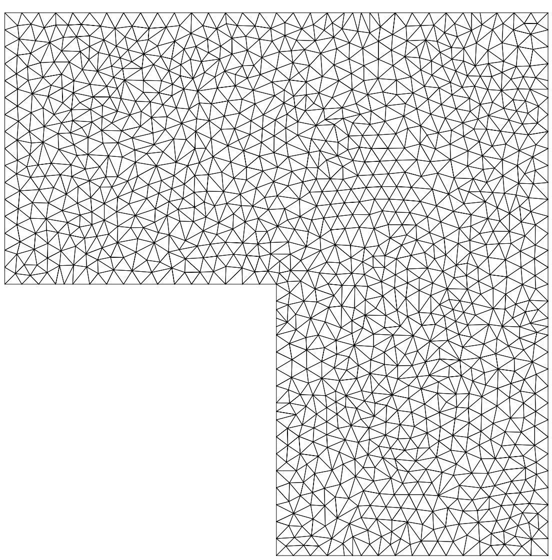

FIGURE 4
Mesh for the test problem (1229 nodes, 2318 elements, $h_{mean} = 0.032$*)*

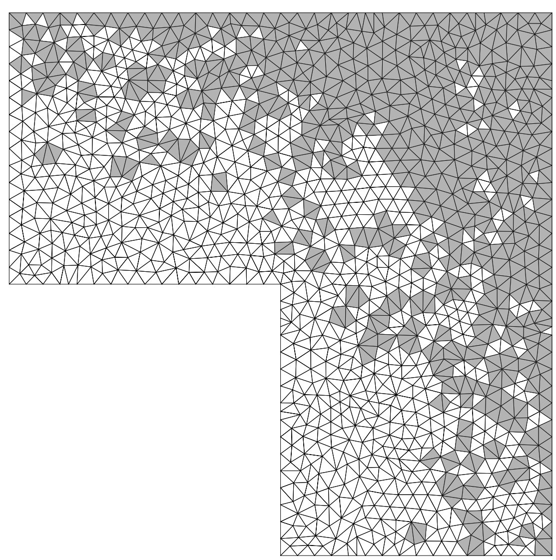

FIGURE 5
Local Peclet number for the test mesh. Shaded triangles are convection-dominated ($Pe_K > 1$*)*

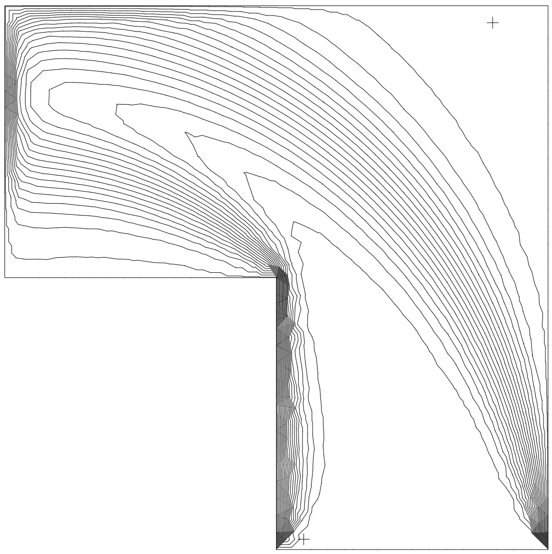

FIGURE 6
SUPG stabilization

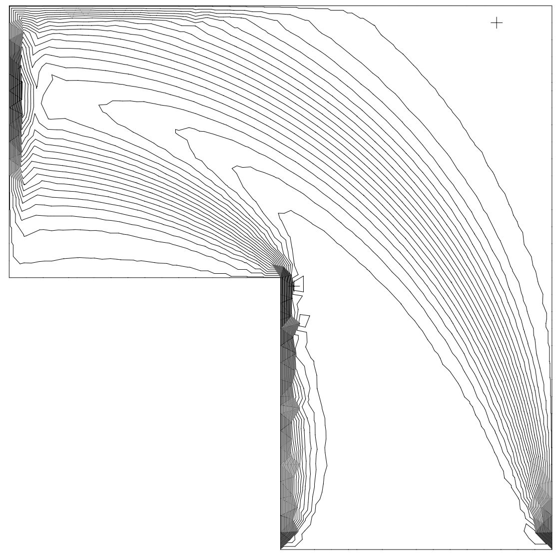

FIGURE 7
Pseudo residual-free bubbles stabilization

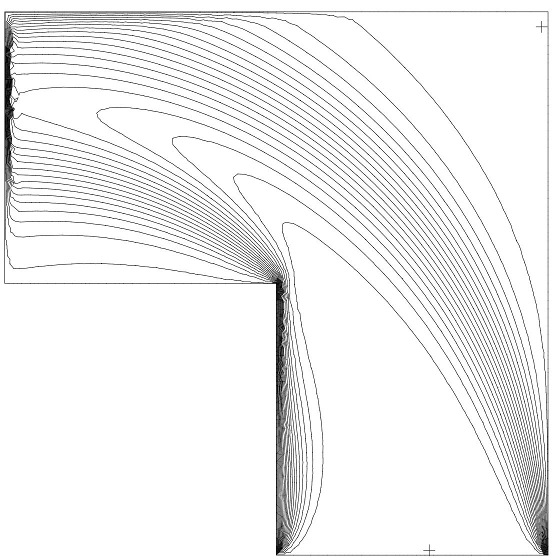

FIGURE 8
Galerkin method (no stabilization) on the refined mesh of Fig. 9

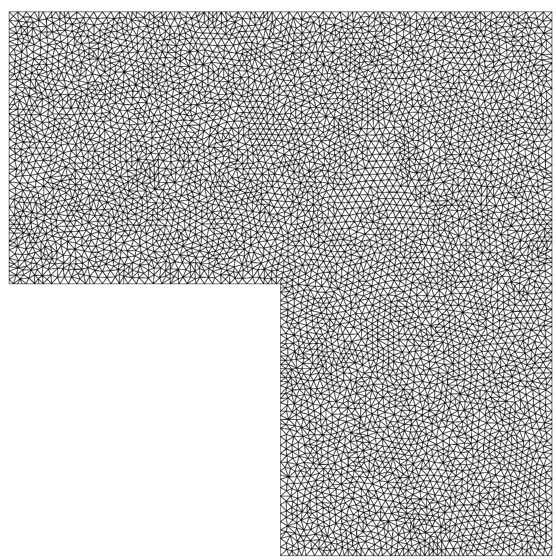

FIGURE 9
Refined mesh (5993 nodes, 11639 elements, $h_{mean} = 0.0014$)