A DOMAIN DECOMPOSITION METHOD FOR BONDED PLATES

G. GEYMONAT

LMT, ENS Cachan/CNRS/ Université Pierre-et-Marie-Curie
61 Av. du Président Wilson, 94235 Cachan (France)
E-mail: geymonat@lmt.ens-cachan.fr

F. KRASUCKI

LMM, CNRS/Université Pierre-et-Marie-Curie
4 place Jussieu, 75252 Paris (France)
E-mail: krasucki@ccr.jussieu.fr

and

D. MARINI

Dipartimento di Matematica and I.A.N.-C.N.R.
Via Abbiategrasso 215,27100 Pavia (Italy)
E-mail: marini@dragon.ian.pv.cnr.it

ABSTRACT

We present a domain decomposition type algorithm for dealing with the numerical solution of bonded plates

1. Introduction

Since a pioneering work by Goland and Reissner in 1944 [6] the bonding of two elastic three dimensional structures by an adhesive layer is treated with asymptotic analysis. (See, e.g., [1],[2],[5],[8].) In the resulting limit problem the adhesive disappears from a geometrical point of view but it gives rise to suitable transmission conditions. In [3] we introduced and analyzed a domain decomposition type procedure to deal with the limit problem numerically. In the present paper we apply the same technique to the bending of two thin elastic plates (Love-Kirchhoff), bonded in their common plane by an adhesive layer. This layer is also treated as a Love-Kirchhoff plate having, in its plane, a small dimension with respect of those of the two adherent plates. Let ε donote the smallness ratio. The type of transmission conditions in the limit problem depends on the ratio of the bending rigidity coefficients. We refer to [4] for the derivation of the limit problem in the different cases. In what follows we shall consider the case where the bending rigidity coefficient of the glue is given by $\varepsilon^3 D_0$, D_0 being of the same order of magnitude of D^+, D^-, the bending coefficients of the adherents.

2. Position of the problem

Let Ω^+ and Ω^- denote the two plates, that we assume to be open connected subsets of $\mathbf{R}^2$ with boundaries $\partial\Omega^+$ and $\partial\Omega^-$ piecewise of class C^2, and let $S = \partial\Omega^+ \cap \partial\Omega^-$ be a non empty regular curve of positive measure. Let Ω be the union of Ω^+ and Ω^-, with boundary $\partial\Omega$, and let $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$, $\Gamma^- = \partial\Omega^- \cap \partial\Omega$. For simplicity, assume that the plate is clamped on $\partial\Omega$. For a function v defined on Ω, let v^+ (resp. v^-) denote the restriction of v to Ω^+

(resp. Ω^-). The local equations are (see [4])

$$\left\{\begin{array}{rcll} D^+\Delta^2 w^+ & = & p^+ & in\ \Omega^+ \\ D^-\Delta^2 w^- & = & p^- & in\ \Omega^- \\ w^+ = \frac{\partial w^+}{\partial n} & = & 0 & on\ \Gamma^+ \\ w^- = \frac{\partial w^-}{\partial n} & = & 0 & on\ \Gamma^- \end{array}\right. \tag{1}$$

with the transmission conditions on S

$$\left\{\begin{array}{rcll} M_n(w^+) = M_n(w^-) & = & 0 & on\ S \\ K_n(w^+) & = & -12D_0(w^+ - w^-) & on\ S \\ K_n(w^-) & = & 12D_0(w^+ - w^-) & on\ S \end{array}\right. \tag{2}$$

where $p^+,\ p^-$ are the applied external loads, $\mathbf{n}^+$ (resp. $\mathbf{n}^-$) is the outward unit normal to Ω^+ (resp. Ω^-), M_n is the normal bending moment, and K_n the normal Kirchhoff shear force. In order to apply a domain decomposition type procedure, we observe that the boundary conditions (2) can be rewritten as

$$\left\{\begin{array}{rcll} M_n(w^+) = M_n(w^-) & = & 0 & on\ S \\ K_n(w^+) & = & -K_n(w^-) & on\ S \\ K_n(w^+) + 24D_0 w^+ & = & K_n(w^-) + 24D_0 w^- & on\ S \end{array}\right. \tag{3}$$

Next, for $g \in L^2(S)$, consider the following problems

$$\left\{\begin{array}{rcll} D^+\Delta^2 w^+ & = & p^+ & in\ \Omega^+ \\ w^+ = \frac{\partial w^+}{\partial n} & = & 0 & on\ \Gamma^+ \\ M_n(w^+) & = & 0 & on\ S \\ K_n(w^+) + 24D_0 w^+ & = & g & on\ S \end{array}\right. \quad \left\{\begin{array}{rcll} D^-\Delta^2 w^- & = & p^- & in\ \Omega^- \\ w^- = \frac{\partial w^-}{\partial n} & = & 0 & on\ \Gamma^- \\ M_n(w^-) & = & 0 & on\ S \\ K_n(w^-) + 24D_0 w^- & = & g & on\ S \end{array}\right. \tag{4}$$

For any given $g \in L^2(S),\ p^+ \in L^2(\Omega^+),\ p^- \in L^2(\Omega^-)$ problems (4) have a unique solution $w^+ \in H^2(\Omega^+)$, and $w^- \in H^2(\Omega^-)$ respectively. (Note that the boundary conditions (4) actually induce more regularity on the solutions.) Due to linearity, these solutions can be split as

$$w^+ = w_p^+ + w_g^+, \qquad w^- = w_p^- + w_g^-, \tag{5}$$

with $w_p^+,\ w_p^-$ solutions of (4) with $g = 0$, and $w_g^+,\ w_g^-$ solutions of (4) with $p^+ = 0,\ p^- = 0$. We can then define the linear continuous operators $T_p^+,\ T_p^-,\ T_g^+,\ T_g^-$

$$\begin{array}{c} p^+ \in L^2(\Omega^+) \longrightarrow w_p^+ = T_p^+(p^+), \quad p^- \in L^2(\Omega^-) \longrightarrow w_p^- = T_p^-(p^-), \\ g \in L^2(S) \longrightarrow w_g^+ = T_g^+(g), \quad w_g^- = T_g^-(g), \end{array} \tag{6}$$

so that (5) becomes

$$w^+ = T_p^+(p^+) + T_g^+(g) \qquad w^- = T_p^-(p^-) + T_g^-(g). \tag{7}$$

Next, let $\mathcal{A}$ be the operator from $L^2(S)$ in itself defined as

$$g \in L^2(S) \longrightarrow \mathcal{A}g = (w_g^+ + w_g^-)_{|S} \equiv (T_g^+(g) + T_g^-(g))_{|S}. \tag{8}$$

It is immediate to check that $\mathcal{A}$ is linear and continuous. Moreover, thanks to the trace theorem (see, e.g., [7]), we have in particular $w^+_{g\,|S} \in H^{3/2}_{00}(S)$, $w^-_{g\,|S} \in H^{3/2}_{00}(S)$, so that $\mathcal{A}$ is linear and continuous from $L^2(S)$ into $H^{3/2}_{00}(S)$.

Going back to formulation (4), note that the continuity condition on K_n in (3) is not taken into account. Hence, we must find a suitable g such that the solutions of (4) verify (3). Since from (4) it follows that $K_n(w^+) + K_n(w^-) \;=\; 2(g - 12D_0(w^+ + w^-))$, such a g will be the solution of the following minimization problem

$$\text{Find } g^* \in L^2(S) \;:\; 0 = J(g^*) \leq J(g) \qquad \forall g \in L^2(S), \tag{9}$$

for the quadratic functional

$$J(g) := ||g - 12D_0(w^+ + w^-)||^2_{0,S}. \tag{10}$$

Using the notation introduced in (7)-(8) we have

$$12D_0(w^+ + w^-)_{|S} \;=\; F + 12D_0\mathcal{A}g, \qquad \text{having set } F := 12D_0(T^+_p(p^+) + T^-_p(p^-))_{|S}, \tag{11}$$

so that (10) can be written as

$$J(g) = ||g - (F + 12D_0\mathcal{A}g)||^2_{0,S}. \tag{12}$$

It is easy to check that $J(g)$ is strictly convex, so that problem (9) has a unique solution g^*, which verifies

$$g^* \;=\; F + 12D_0\mathcal{A}g^*. \tag{13}$$

In order to write the variational formulation of (4) we set

$$V^+ := \{v \in H^2(\Omega^+)\ , v = \partial v/\partial n = 0 \text{ on } \Gamma^+\}, \tag{14}$$
$$V^- := \{v \in H^2(\Omega^-)\ , v = \partial v/\partial n = 0 \text{ on } \Gamma^+\}, \tag{15}$$

$$a^+(v,w) = D^+\int_{\Omega^+} (v_{/11}w_{/11} + 2(1-\nu)v_{/12}w_{/12} + v_{/22}w_{/22} + \nu(v_{/11}w_{/22} + v_{/22}w_{/11})\,dx \tag{16}$$
$$a^-(v,w) = D^-\int_{\Omega^-} (v_{/11}w_{/11} + 2(1-\nu)v_{/12}w_{/12} + v_{/22}w_{/22} + \nu(v_{/11}w_{/22} + v_{/22}w_{/11})\,dx \tag{17}$$

$$\mathcal{a}^+(w,v) \;= a^+(w,v) + 24D_0\textstyle\int_S v\,w\,ds, \tag{18}$$
$$\mathcal{a}^-(w,v) \;= a^+(w,v) + 24D_0\textstyle\int_S v\,w\,ds. \tag{19}$$

The variational formulation of problems (4) is then

$$\left\{\begin{array}{l} \text{Find } w^+ \in V^+ \text{ such that :} \\ \mathcal{a}^+(w^+,v) \;=\; (p^+,v) + (g,v)_S \qquad \forall v \in V^+, \end{array}\right. \tag{20}$$

$$\left\{\begin{array}{l} \text{Find } w^- \in V^- \text{ such that :} \\ \mathcal{a}^-(w^-,v) \;=\; (p^-,v) + (g,v)_S \qquad \forall v \in V^-. \end{array}\right. \tag{21}$$

Existence, uniqueness and a-priori error bounds for the solutions of (20)-(21) are ensured by the continuity and coercivity properties of the bilinear forms $a^+,\ a^-$.

3. The Algorithm

We shall now present a domain decomposition type algorithm, based on the variational formulations (20)-(21) and the minimum problem (9), for which we shall prove convergence. Compute $w_p^+ = T_p^+(p^+),\ w_p^- = T_p^-(p^-)$ solutions of

$$\begin{aligned} w_p^+ \in V^+ :\quad a^+(w_p^+, v) &= (p^+, v) \qquad \forall v \in V^+, & (22)\\ w_p^- \in V^- :\quad a^-(w_p^-, v) &= (p^-, v) \qquad \forall v \in V^-, & (23) \end{aligned}$$

and set

$$g^0 \ = \ 12D_0(w_p^+ + w_p^-)_{|S} \ (= F). \qquad (24)$$

For $m \geq 0$ compute the solutions $w_m^+ = T_g^+(g^m)$, $w_m^- = T_g^-(g^m)$ of the problems

$$\begin{aligned} w_m^+ \in V^+ \ :\ a^+(w_m^+, v) &= (g^m, v)_S \qquad \forall v \in V^+, & (25)\\ w_m^- \in V^- \ :\ a^-(w_m^-, v) &= (g^m, v)_S \qquad \forall v \in V^-. & (26) \end{aligned}$$

Then set

$$\begin{aligned} \tilde{g}^m &:= g^m - 12D_0(w_m^+ + w_m^-)_{|S}, & (27)\\ g^{m+1} &:= g^m - \rho(\tilde{g}^m - g^0), & (28) \end{aligned}$$

and compute the solutions w_{m+1}^-, w_{m+1}^+ of (25)-(26) with the new datum g^{m+1}. In (28) $\rho > 0$ is a parameter to be chosen in order to have convergence of g^m to g^*, as $m \to \infty$, where g^* is defined in (13). In order to prove convergence we shall use the following result

Theorem 1 *$\mathcal{A}$ is a compact operator. Moreover, the eigenvalues z of $12D_0\mathcal{A}$ are all real and verify*

$$\exists\, C_1 > 0 \ \text{ such that } \qquad 0 \leq z \leq 1 - C_1 < 1 \qquad \forall z. \qquad (29)$$

Proof The proof is a slight modification of that given in [3] and we shall not report it here. We can now prove the following convergence theorem.

Theorem 2 *There exists a $\rho_0 \geq 1$ such that, for $\rho \in]0, \rho_0[$ we have*

$$\lim_{m\to\infty} g^m = g^*, \qquad (30)$$

where g^m is the sequence defined in (22)-(28), and g^* is defined in (13).

Proof Note that, according to definition (8), (27) can be rewritten as

$$\tilde{g}^m \ = \ (I - 12D_0\mathcal{A})g^m. \qquad (31)$$

From (28) and (31), using (24) and (13) we then have

$$\begin{aligned} g^{m+1} - g^* \quad &= \ (1-\rho)g^m + \rho 12D_0\mathcal{A}g^m + \rho g^0 - g^* + \rho g^* - \rho g^* \\ &= (1-\rho)g^m + \rho 12D_0\mathcal{A}g^m + \rho(g^0 - g^*) - (1-\rho)g^* \\ &= (1-\rho)(g^m - g^*) + \rho 12D_0\mathcal{A}(g^m - g^*) \\ &= ((1-\rho)I + \rho 12D_0\mathcal{A})(g^m - g^*). \end{aligned} \qquad (32)$$

Recursive application of (32) yields

$$g^{m+1} - g^* = ((1-\rho)I + \rho 12 D_0 \mathcal{A})^{m+1}(g^0 - g^*), \quad \text{with } g^0 - g^* = -12 D_0 \mathcal{A} g^*. \tag{33}$$

Convergence will be proved if we can show that

$$\lim_{m\to\infty} |||((1-\rho)I + \rho 12 D_0 \mathcal{A})^{m+1}||| = 0, \tag{34}$$

where $|||L|||$ denotes the norm of the operator L. From a theorem by Gelfand, if L is bounded then $\lim_{n\to\infty} |||L^n|||^{1/n} = \sup\{|\lambda|,\ \lambda \in \sigma(L)\}$, $\sigma(L)$ being the spectrum of L. Thanks to Theorem 1, the spectrum of the operator $(1-\rho)I + \rho 12 D_0 \mathcal{A}$ is given by $1-\rho$ and

$$\lambda_j = (1-\rho) + \rho z_j, \tag{35}$$

z_j being the eigenvalues of $12 D_0 \mathcal{A}$. Proving (34) amounts then to prove that

$$f(\rho) := \max\{|1-\rho|, \max_j |\lambda_j|\} < 1, \tag{36}$$

and this is true for all $\rho \in]0, 2[$, since the inequality

$$-2 < \rho(z_j - 1) < 0 \qquad \forall j \tag{37}$$

is verified for all the values $\rho \in]0, \frac{2}{1-z_{min}}[$ and $\frac{2}{1-z_{min}} \geq 2$. ∎

Remark The optimal value for ρ is the minimizing argument of the function $f(\rho)$ in (36). A simple computation gives $\rho_{opt} = \frac{2}{2-z_{max}} > 1$.

4. References

1. P. Destuynder, F. Michavila, A. Santos and Y. Ousset, Some theoretical aspects in computational analysis of adhesive lap joints, *Internat. J. Numer. Methods Engrg.* **35** (1992) 1237–1262.
2. G. Geymonat, F. Krasucki and S. Lenci, Analyse asymptotique du comportement d'un assemblage collé, *C.R. Acad. Sci. Paris* **322, Série I** (1996) 1107–1112.
3. G. Geymonat, F. Krasucki, D. Marini and M. Vidrascu (to appear)
4. G. Geymonat and F. Krasucki, (to appear)
5. Y. Gilibert and A. Rigolot, Analyse asymptotique des assemblages collés à double recouvrement sollicités au cisaillement en traction, *J. Mec. Appl.* **3** (1979) 341–372.
6. M. Goland and E. Reissner, The stresses in cemented joints, *J. Appl. Mech. ASME* **11** (1944) A.17–A.27.
7. P. Grisvard, *Elliptic problems in nonsmooth domains* (Pitman, London) (1985).
8. A. Klarbring, Derivation of a model of adhesively bonded joints by the asymptotic expansion method, *Int. J. Engrg. Sci.* **29** (1991) 493–512.