A PRECONDITIONER FOR DOMAIN DECOMPOSITION METHODS FOR ADVECTION-DOMINATED PROBLEMS

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Abstract

We present a new preconditioner for the Schur complement in domain decomposition methods. The new preconditioner seems to be particularly suited for dealing with advection-dominated elliptic problems.

1 INTRODUCTION

A common procedure in domain decomposition methods is to split the domain Ω, where the differential problem is posed, into subdomains (or macro elements) Ω_k. The unknowns internal to each subdomain are then eliminated and one is left with a system of the type

$$\mathcal{S}_h\psi_h \; = \; g_h \tag{1.1}$$

in the unknowns ψ_h lying on the interfaces between subdomains. This kind of procedure is particularly suited for the use of parallel computers, as the elimination of the internal variables is typically done independently in each subdomain. The operator $\mathcal{S}_h$, commonly called Schur complement in the

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domain decomposition terminology, can be seen as the discretization of a pseudo-differential operator of order 1 acting on $\Sigma =$ union of the interfaces between subdomains (see, e.g., [1]-[2].) For an efficient solution of (1.1) it is then necessary to have a good preconditioner. As already pointed out in [1], an ideal preconditioner should behave as an operator of order -1, which is not so easy to obtain in practice. However, several preconditioners have been proposed in the literature (see, e.g., [3] and the references therein), although most of them were restricted to symmetric problems. In [2] we suggested the use of the following preconditioner

$$\mathcal{S}_h^T D_h^{-1} \mathcal{S}_h \psi_h \ = \ \mathcal{S}_h^T D_h^{-1} g_h \ , \tag{1.2}$$

where D_h is a discrete second tangential derivative operator defined by

$$< D_h \psi, \phi >= \int_\Sigma \psi_{/t} \phi_{/t} \, ds \ , \tag{1.3}$$

and $\phi_{/t}$ is the tangential derivative of ϕ. Preconditioner (1.2) balances an operator of order -2 (D_h^{-1}) with two operators of order 1 ($\mathcal{S}_h$ and $\mathcal{S}_h^T$), and it has the advantage of giving rise to a symmetric problem even if the original differential problem (and hence $\mathcal{S}_h$) is not symmetric. However, for strongly advection-dominated problems, as those arising in computational fluid dynamics, this preconditioner stops being effective. For this reason we propose a variant of (1.2) of the following type

$$\mathcal{S}_h^T D_{h,\omega}^{-1} \mathcal{S}_h \psi_h \ = \ \mathcal{S}_h^T D_{h,\omega}^{-1} g_h \ , \tag{1.4}$$

with $D_{h,\omega}$ given by

$$< D_{h,\omega} \psi, \phi >= \int_\Sigma \left(\psi_{/t} \phi_{/t} + \omega \psi \phi \right) \, ds \ . \tag{1.5}$$

In the experiments carried out so far on the model problem

$$\begin{cases} -\Delta u + \beta u_x \ = \ f & \text{in } \Omega, \\ \qquad\qquad u \ = \ g & \text{on } \partial\Omega \end{cases} \tag{1.6}$$

and on a structured grid it appears that a good choice for ω is given by: $\omega \simeq \sqrt{\beta}$ on the vertical edges (cross-wind), and $\omega = 0$ on the horizontal edges (streamline). As it can be seen in Sect.3, the new preconditioner proved to be extremely effective, at least in the cases tested so far. Nevertheless we believe that much work is still to be done in order to understand its behaviour and to optimize it in cases of practical relevance.

An outline of the paper is as follows. In Sect. 2 we recall the three-field formulation introduced in [4] which leads, in a simple and natural way, to the continuous version $\mathcal{S}$ of the Schur complement and to its discrete counterpart $\mathcal{S}_h$. In Sect. 3 we describe the preconditioner and present numerical results obtained on an nCUBE2 parallel computer.

2 THE THREE-FIELD FORMULATION

For the convenience of the reader we recall the three-field formulation introduced in [4] for linear second order elliptic operators. Let $\Omega \subset \mathbf{R}^2$ be a polygonal domain split into a finite number of polygonal subdomains Ω_k $(k = 1, .., K)$ (see fig. 1):

$$\Omega = \overline{\bigcup_k \overset{\circ}{\Omega}_k}, \tag{2.1}$$

and define

$$\Gamma_k = \partial\Omega_k \quad ; \quad \Sigma = \bigcup_k \Gamma_k. \tag{2.2}$$

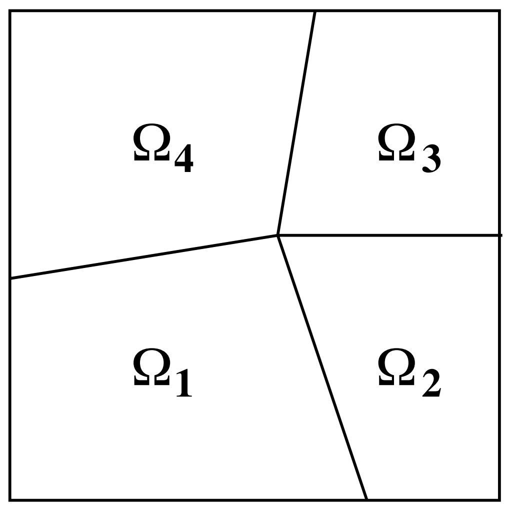

Fig. 1 Example of subdivision

Let A be a linear elliptic operator of the form

$$Au = \sum_i \{\sum_j (-\frac{\partial}{\partial x_j}(a_{ij}(x)\frac{\partial u}{\partial x_i} + b_j(x)u) + c_i(x)\frac{\partial u}{\partial x_i}\} + d(x)u. \tag{2.3}$$

We assume that the coefficients a_{ij}, b_j, c_i, d belong to $L^\infty(\Omega)$ and are smooth in each Ω_k, and we consider the bilinear forms associated with A in each Ω_k, that is,

$$for\ u,\, v \in\ H^1(\Omega_k):$$
$$a_k(u,\, v) := \int_{\Omega_k} \{\sum_i (\sum_j (a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + b_j u \frac{\partial v}{\partial x_j}) + c_i \frac{\partial u}{\partial x_i} v) + duv\}\, dx. \tag{2.4}$$

We also set, for $u,\, v \in \prod_k H^1(\Omega_k)$

$$a(u,\, v) := \sum_k a_k(u,\, v); \tag{2.5}$$

for the sake of simplicity we also assume that there exists a constant $\alpha > 0$ such that

$$a(v,\, v) \geq \alpha ||v||^2_{H^1(\Omega)} \qquad \forall v \in H^1_0(\Omega). \tag{2.6}$$

In what follows we shall use the following notation: $(.,.)$ will be the usual inner product in $L^2(\Omega)$; for $k = 1, ...K$, $(.,.)_k$ will be the inner product in $L^2(\Omega_k)$ and $< .,. >_k$ will be the inner product in $L^2(\Gamma_k)$ (or, when necessary, the duality pairing between $H^{-\frac{1}{2}}(\Gamma_k)$ and $H^{\frac{1}{2}}(\Gamma_k)$). Let us now introduce the spaces that will be used in our macro-hybrid formulation. For $k = 1, ..., K$ we set

$$V_k := H^1(\Omega_k), \tag{2.7}$$
$$M_k := H^{-\frac{1}{2}}(\Gamma_k). \tag{2.8}$$

We then define

$$V := \prod_k V_k, \tag{2.9}$$
$$M := \prod_k M_k, \tag{2.10}$$

and

$$\Phi := \{\phi \in L^2(\Sigma)\ :\ \exists v \in H^1_0(\Omega) \text{ with } \phi = v_{|\Sigma}\} \equiv H^1_0(\Omega)_{|\Sigma}. \tag{2.11}$$

For every f, say, in $L^2(\Omega)$, we can now consider the following two problems:

$$\begin{cases} \text{find } w \in H^1_0(\Omega) \text{ such that} \\ a(w, v) = (f, v) \qquad \forall v \in H^1_0(\Omega) \end{cases} \tag{2.12}$$

and

$$\begin{cases} \text{find } u \in V,\ \lambda \in M \text{ and } \psi \in \Phi \text{ such that} \\ \text{i) } a(u, v) - \sum_k < \lambda^k, v^k >_k\ =\ (f, v) \qquad \forall v \in V \\ \text{ii) } \sum_k < \mu^k, \psi - u^k >_k = \ 0 \qquad \forall \mu \in M \\ \text{iii) } \sum_k < \lambda^k, \phi >_k = 0 \qquad \forall \phi \in \Phi. \end{cases} \tag{2.13}$$

Theorem For every $f \in L^2(\Omega)$, both problems (2.12) and (2.13) have a unique solution. Moreover, we have

$$u^k = w \qquad \text{in } \Omega_k \quad (k = 1, ..., K), \tag{2.14}$$

$$\lambda^k = \frac{\partial w}{\partial n_A^k} \qquad \text{on } \Gamma_k \quad (k = 1, ..., K), \tag{2.15}$$

$$\psi = w \qquad \text{on } \Sigma \tag{2.16}$$

where $\partial w / \partial n_A^k$ is the outward conormal derivative (of the restriction of w to Ω_k) with respect to the operator A.

Proof For the proof we refer to [4]. ∎

It is very important, for applications to domain decomposition methods, to remark explicitly that the first two equations of (2.13) can be written as

$$\begin{cases} a_k(u^k, v^k) - < \lambda^k, v^k >_k = (f, v^k) & \forall v^k \in V_k, \quad \forall k \\ < \mu^k, u^k >_k = < \psi, \mu^k >_k & \forall \mu^k \in M_k, \quad \forall k. \end{cases} \tag{2.17}$$

In particular, for all fixed k, assuming f and ψ as data, (2.17) is the variational formulation of the Dirichlet problem

$$\begin{cases} Au^k = f & \text{in } \Omega_k, \\ u^k = \psi & \text{on } \Gamma_k, \end{cases} \tag{2.18}$$

where the boundary condition is imposed by means of a Lagrange multiplier (that eventually comes out to be $\lambda^k \equiv \partial u^k / \partial n_A^k$) as in Babuška [5]. Hence, for f and ψ given, the resolution of the first two equations of (2.13) amounts to the resolution of K independent Dirichlet problems. In operator form (2.13) can be written as

$$\begin{cases} Au - B\lambda = f \\ -B^T u + C\psi = 0 \\ C^T u = 0 \end{cases} \tag{2.19}$$

where, as already noticed, the operator

$$\mathcal{A} := \begin{pmatrix} A & -B \\ -B^T & 0 \end{pmatrix} \tag{2.20}$$

is "block diagonal" in $V \times M$ and invertible. Setting now

$$\mathcal{C}^T(u, \lambda) := C^T \lambda, \tag{2.21}$$

$$(u_f, \lambda_f) := \mathcal{A}^{-1}(f, 0), \tag{2.22}$$

$$g := \mathcal{C}^T(u_f, \lambda_f) = C^T \lambda_f, \tag{2.23}$$

problem (2.13) can now be written as

$$\mathcal{C}^T \mathcal{A}^{-1} \mathcal{C} \psi \;=\; g. \tag{2.24}$$

Setting

$$\mathcal{S} := \mathcal{C}^T \mathcal{A}^{-1} \mathcal{C}, \tag{2.25}$$

the problem is now

$$\mathcal{S}\psi \;=\; g. \tag{2.26}$$

Remark Notice that, in the usual language of domain decomposition methods, $\mathcal{S}$ is the Poincaré-Steklov operator on Σ, associated with the elliptic operator A. We also notice that the dual operator $\mathcal{S}^T$ (that will be used in the sequel) can be obtained by an identical procedure starting from the adjoint problem. More precisely, to get $\mathcal{S}^T$ we use in (2.13) $a^T(u, v) := a(v, u)$ instead of $a(u, v)$, and we repeat the procedure (2.19)-(2.26) with A^T instead of A. ■

Problem (2.13) can now be approximated in many different ways. Choosing $V_h,\ M_h$, and Φ_h finite dimensional subspaces of $V,\ M,\ \Phi$, we can consider the discretized problem

$$\begin{cases} \text{find } u_h \in V_h,\ \lambda_h \in M_h \text{ and } \psi_h \in \Phi_h \text{ such that} \\ \text{i) } a(u_h, v) - \sum_k < \lambda_h^k, v^k >_k \;=\; (f, v) \qquad \forall v \in V_h \\ \text{ii) } \sum_k < \mu^k, \psi_h - u_h^k >_k = \; 0 \qquad \forall \mu \in M_h \\ \text{iii) } \sum_k < \lambda_h^k, \phi >_k = 0 \qquad \forall \phi \in \Phi_h. \end{cases} \tag{2.27}$$

It is clear that suitable inf-sup conditions should be assumed on the choice of $V_h,\ M_h$ and Φ_h, unless the formulation is properly stabilized as in [6] and [7]. We shall not address these questions here. We point out, instead, that, if one takes a finite element approximation $\tilde{V}_h$ of $H^1(\Omega)$, on a mesh compatible with the decomposition (2.1), one can set

$$\begin{cases} V_h^k := \tilde{V}_{h|\Omega_k} & ; \quad V_h = \prod V_h^k; \\ \tilde{V}_h^\circ = \tilde{V}_h \bigcap H_0^1(\Omega_k) & ; \quad \Phi_h = \tilde{V}_{h|\Sigma}^\circ; \\ M_h^k := (V_{h\ |\Gamma_k}^k)' & ; \quad M_h := \prod M_h^k. \end{cases} \tag{2.28}$$

It is easy to check that with these choices the solution of (2.27) is nothing else but the standard finite element approximation of the solution of (2.13) by means of the subspace $\tilde{V}_h^\circ$. Moreover, the discrete analogue $\mathcal{S}_h$ of $\mathcal{S}$ is the classical Schur complement, and the discrete analogue $\mathcal{S}_h^T$ of $\mathcal{S}^T$ is, at the same time, the transpose of the Schur complement $\mathcal{S}_h$ and the Schur complement of the transpose problem. In the next section we shall concentrate our attention on discretizations of this type, and present a new preconditioner for problem

$$\mathcal{S}_h \psi_h \;=\; g_h \tag{2.29}$$

which is, with obvious notation, the finite element discretization of (2.26).

3 THE PRECONDITIONER

In [2] we suggested the use of the following preconditioner for solving the system (2.29)

$$\mathcal{S}_h^T D_h^{-1} \mathcal{S}_h \psi_h \;=\; \mathcal{S}_h^T D_h^{-1} g_h, \tag{3.1}$$

where D_h is a discrete second tangential derivative operator defined on Φ_h by

$$< D_h \psi, \phi >= \int_\Sigma \psi_{/t} \phi_{/t}\, ds \qquad\qquad \phi,\ \psi \in \Phi_h, \tag{3.2}$$

where $\phi_{/t}$ is the tangential derivative of ϕ. Extensive numerical results showed that preconditioner (3.1) is very effective for symmetric problems, as well as for moderately unsymmetric ones. Its performance deteriorates when the number of subdomains grows. In particular, in agreement with the analysis of [1], the quality deteriorates linearly in $\sqrt{N}$, if N is the number of subdomains. In Table 1 we report the results obtained for the model problem

$$\begin{cases} -\Delta u \;=\; f & \text{in } \Omega, \\ \quad\; u \;=\; g & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

with $\Omega =$ unit square subdivided into N squares as in fig. 2. For simplicity, results where obtained with $f = 0,\ g = 1$, taking as initial guess for ψ_h a random generated vector. In Table 1 we report the number of conjugate gradient iterations necessary to reduce the maximum norm of the initial residual by a factor 10^{-7}. The mesh size h in each subdomain varies from 1/10 to 1/100.

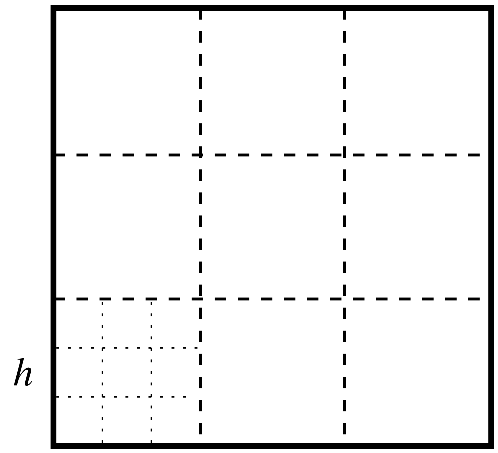

Fig. 2 Example of subdivision in $N = 9$ subdomains

h	N=4	N=100
1/10	12	48
1/20	13	50
1/30	13	51
1/40	14	51
1/50	14	51
1/60	14	52
1/70	14	52
1/80	15	53
1/90	15	53
1/100	15	53

Table 1

As an example of a nonsymmetric problem let us consider now

$$\begin{cases} -\Delta u + \beta u_x &= f \qquad \text{in } \Omega, \\ u &= g \qquad \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

with Ω, f, g as in problem (3.3) and β constant. For $\beta >> 1$ we are then simulating an advection-dominated problem on a structured grid. Problem (3.4) was discretized with linear conforming finite elements, upwinded for the treatment of the advective term to give a backward finite difference scheme. For values of β not too big, preconditioner (3.1) is still quite effective. However, for big values of β, that is, for strongly advection-dominated problems, the quality of preconditioner (3.1) deteriorates till a complete failure. (See

Tables 2-a and 2-b, where the results obtained for subdivisions into N =4 and N =16 subdomains are shown. 'Fail' means that more than 100 iterations were required to reduce the maximum norm of the initial residual by a factor 10^{-7}.)

β	$h = \frac{1}{10}$	$h = \frac{1}{20}$	$h = \frac{1}{30}$	$h = \frac{1}{40}$
10	13	14	14	14
10^2	19	24	25	29
10^3	22	32	41	47
10^4	23	36	49	58

Table 2-a - N=4

β	$h = \frac{1}{10}$	$h = \frac{1}{20}$	$h = \frac{1}{30}$	$h = \frac{1}{40}$
10	25	28	30	31
10^2	56	67	81	84
10^3	99	Fail	Fail	Fail

Table 2-b - N=16

We propose therefore the following modification:

$$\mathcal{S}_h^T D_{h,\omega}^{-1} \mathcal{S}_h \psi_h \;=\; \mathcal{S}_h^T D_{h,\omega}^{-1} g_h \tag{3.5}$$

whith $D_{h,\omega}$ given by

$$< D_{h,\omega}\psi, \phi >= \int_\Sigma \left(\psi_{/t}\phi_{/t} + \omega\psi\phi\right) \, ds \; . \tag{3.6}$$

A good value for the parameter ω is given, experimentally, by: $\omega \simeq \sqrt{\beta}$ on the vertical edges (cross-wind), and $\omega = 0$ on the horizontal edges (streamline). Tables 3-a and 3-b report the results for subdivisions into 4 and 16 subdomains respectively.

At last, in Table 4 the results obtained solving the nonsymmetric problem (3.4) with $f = 0$ and $g(x, y) = 1 - x$ (so that the solution has boundary layers) are reported.

β	ω_v	$h=\frac{1}{10}$	$h=\frac{1}{20}$	$h=\frac{1}{30}$	$h=\frac{1}{40}$
10^4	10^2	19	19	27	35
10^5	316.2	10	12	17	17
10^6	10^3	10	10	10	12
10^7	3162.3	8	8	10	10

Table 3-a - N=4

β	ω_v	$h=\frac{1}{10}$	$h=\frac{1}{20}$	$h=\frac{1}{30}$	$h=\frac{1}{40}$
10^3	31.6	63	72	64	72
10^5	316.2	31	36	52	62
10^7	3162.3	22	23	25	29

Table 3-b - N=16

N	β	ω_v	$h=\frac{1}{10}$	$h=\frac{1}{20}$	$h=\frac{1}{30}$	$h=\frac{1}{40}$
4	10^7	3162.3	8	8	10	10
16	10^7	3162.3	22	23	29	29

Table 4

We can see that the lack of regularity of the solution does not affect the overall quality of the convergence rate. The new preconditioner looks then quite interesting. An analysis of its behaviour, as well as extensive numerical results, will be presented in forthcoming papers.
All the results up to 16 subdomains where obtained on a parallel distributed memory computer nCUBE2 mod. 6401, with 16 processor elements, of the Istituto di Analisi Numerica del C.N.R. . For more than 16 subdomains an nCUBE2 with 128 processors of CNUCE was used. In both cases we used a parallel code, obtained devoting each processor element to the solution of each subproblem (on each subdomain), whereas the solution of each subproblem was carried out in a sequential mode. The values of the classical measures commonly used for evaluating the performance of a parallel algorithm (*speedup* and *efficiency*) were examined and they looked quite satisfactory.

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